On Positivity of Polynomials: The Dilation Integral Method

B. Ross Barmish, *Fellow, IEEE*, Pavel S. Shcherbakov, Sheila R. Ross, *Member, IEEE*, and Fabrizio Dabbene, *Member, IEEE*

***Abstract*—The focal point of this paper is the well known problem of polynomial positivity over a given domain. More specifically, we consider a multivariate polynomial $f(x)$ with parameter vector x restricted to a hypercube $\mathcal{X} \subset \mathrm{R}^n$. The objective is to determine if $f(x) > 0$ for all $x \in \mathcal{X}$. Motivated by NP-Hardness considerations, we introduce the so-called *dilation integral method*. Using this method, a "softening" of this problem is described. That is, rather than insisting that $f(x)$ be positive for all $x \in \mathcal{X}$, we consider the notions of *practical positivity* and *practical non-positivity*. As explained in the paper, these notions involve the calculation of a quantity $\epsilon > 0$ which serves as an upper bound on the percentage volume of violation in parameter space where $f(x) \leq 0$. Whereas checking the polynomial positivity requirement may be computationally prohibitive, using our ϵ-softening and associated *dilation integrals*, computations are typically straightforward. One highlight of this paper is that we obtain a sequence of upper bounds ϵ_k which are shown to be "sharp" in the sense that they converge to zero whenever the positivity requirement is satisfied. Since for fixed n, computational difficulties generally increase with k, this paper also focuses on results which reduce the size of the required k in order to achieve an acceptable percentage volume certification level. For large classes of problems, as the dimension of parameter space n grows, the required k value for acceptable percentage volume violation may be quite low. In fact, it is often the case that low volumes of violation can be achieved with values as low as $k = 2$.**

***Index Terms*—Approximation methods, integration, numerical analysis, polynomials, risk analysis, robustness, uncertain systems.**

I. Introduction

THIS paper is motivated by the following general problem which lies at the heart of a voluminous body of literature dealing with robustness of systems: *Given a multivariate polynomial $f(x)$ in the vector variable $x = (x_1, x_2, \ldots, x_n)$, determine if the inequality*

$$f(x) > 0$$

is satisfied for all x in a given set $\mathcal{X}$. This being the case, we call $(f, \mathcal{X})$ a positive pair. For the case when the weaker condition $f(x) \geq 0$ is guaranteed, we call $(f, \mathcal{X})$ a non-negative pair.

With these specific problems being the focal point of this paper, it is also noted that the theory to follow can readily be modified to address the case of multiple inequalities $f_i(x) > 0$.

A. Examples

To provide a simple illustration showing how a typical robustness problem is massaged into a polynomial positivity problem, we begin with a three-dimensional single-input state space pair (A_0, b_0) which is assumed to be controllable. That is, the nominal controllability matrix

$$\mathcal{C}_0 = [\, b_0 \quad A_0 b_0 \quad A_0^2 b_0 \,]$$

is nonsingular. In this context, the question of *robust controllability* arises when uncertain parameters x_i are introduced. To illustrate, suppose $|x_i| \leq 1$ describes the admissible values of the ith uncertain parameter with resulting space pair $(A(x), b(x))$ described by

$$A(x) = A_0 + x_2 A_1 + x_1 x_2 A_2 + x_1 x_2 x_3 A_3;$$
$$b(x) = b_0 + x_1 b_1 + x_2 x_3 b_2 + x_1 x_2 b_3$$

where A_1, A_2, A_3 are fixed 3×3 matrices and b_1, b_2, b_3 are fixed 3×1 vectors representing the so-called *uncertainty structure*. Now, to determine if this uncertain system is robustly controllable, we need to check if the controllability matrix

$$\mathcal{C}(x) = [\, b(x) \quad A(x)b(x) \quad A^2(x)b(x) \,]$$

has rank three for all x in the hypercube

$$\mathcal{X} = \{x : |x_i| \leq 1\}.$$

To massage this problem to the one considered in this paper, without loss of generality, we first assume that

$$\det \mathcal{C}_0 > 0.$$

Next, we simply define the polynomial

$$f(x) = \det \mathcal{C}(x)$$

and claim that robust controllability is equivalent to positivity of the pair $(f, \mathcal{X})$. To justify this claim, a simple continuity argument will suffice. That is, since $f(0) > 0$, continuity of $f(x)$ dictates the only way that controllability can be lost is if

Manuscript received July 20, 2007; revised March 29, 2008. Current version published May 13, 2009. This work was supported by the National Science Foundation under Grant ECS-0638816, by the Consiglio Nazionale delle Ricerche (CNR) of Italy under tje RSTL Grant, and through the joint bilateral project between IEIIT-CNR and the Institute of Control Science at the Russian Academy of Sciences (RAS). Recommended by Guest Editors G. Chesi and D. Henrion.

B. R. Barmish is with the Department of Electrical and Computer Engineering, University of Wisconsin, Madison, WI 53706-1691 USA (e-mail: barmish@engr.wisc.edu).

P. S. Shcherbakov is with the Institute of Control Science, Moscow 117997, Russia (e-mail: sherba@ipu.ru).

S. R. Ross is with the Department of Electrical Engineering and Computer Science, Milwaukee School of Engineering, Milwaukee, WI, 53202-3109 USA (e-mail: ross@msoe.edu).

F. Dabbene is with the IEIIT—CNR, Torino 10129, Italy (e-mail: fabrizio.dabbene@polito.it).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2009.2017115

$f(x^0) = 0$ for some $x^0 \in \mathcal{X}$. Later in this paper, this system will be revisited in order to demonstrate the efficacy of the dilation integral method which we put forward.

Using ideas similar to those described in the example above, one can address many unsolved problems from the systems and control literature. For example, while there are now a number of robust stability results in the literature which apply to systems having a so-called linear uncertainty structure, results for the nonlinear case have been slow in forthcoming; see Section III-E for further discussion.

B. Semi-Definite Programming Approach

While not considered here, other papers in this issue provide results which are sparked by some important connections which have been made between the seminal work of [5]–[7] on positive polynomials in the eighties and the solution to robust control problems via semi-definite programming. Taking off on these papers, [8]–[10] introduce the important idea that after reformulating a polynomial positivity problem as a *sum of squares* (SOS) problem, one could obtain a solution via a sequence of semi-definite programs. This meant that convex programming tools could now be brought into play; see [11] for a nice collection of recent developments and a more detailed review of the polynomial positivity literature going back to a conjecture by Hilbert.

C. NP-Hardness Considerations

As motivation for the methods presented in this paper, it is noted that the polynomial positivity problem is NP-Hard; e.g., see the survey of these issues in [12] and various approaches to special cases such as Tarski decision methods in [13], quantifier elimination in [14], the use of Bernstein polynomials as in [15], the use of the structured singular value as in [16], the use of zero set theory in [17], and the use of Gröbner bases as in [18].

D. Key Ideas in This Paper

With nonlinear parameter dependence and computational difficulty serving as motivation, the technical novelty of this paper involves a "softening" of the polynomial positivity problem. To this end, instead of insisting that $f(x)$ be positive for all $x \in \mathcal{X}$, we allow for the possibility that this requirement be violated on a set in parameter space which is deemed to be suitably small. For example, in some robustness problems, if performance specifications are satisfied over all but one percent of the volume of the uncertain parameter set, one may take the point of view that this defines a suitably small risk. More generally, instead of using one percent, the user pre-specifies an acceptable percentage volume of violation $\epsilon > 0$ and computations can be terminated when this risk level is certified. In the sequel, this notion is called "practical positivity" and we can also include a corresponding notion of "practical non-positivity." That is, we show that when the desired volume certification fails to be attained, it follows that the desired inequality $f(x) > 0$ is "close" to being violated.

There are a few important points to note regarding the dilation integral method in this paper versus the semi-definite programming (SDP) approach pursued by other authors in this issue. First, we draw the reader's attention to the fact that in this paper, we have assumed that the restraint set $\mathcal{X}$ is a hypercube[1]. In contrast, the SDP approach allows for more general constraint sets such as polynomial inequalities. Second, for the large class of robustness problems with $\mathcal{X}$ being hypercubic, such as those with interval bounds on uncertain parameters, our framework has some advantages over SDP programming. Namely, as seen in Section I-E, our theory is extremely easy to use and does not require expertise in SDP related areas such as convex programming and linear matrix inequalities. In its simplest form, our method involves generating a function of a scalar variable and picking out its minimum from a plot. While both the SDP approach and the dilation integral approach run into the NP-hardness barrier as the dimension increases, anecdotal evidence given in this paper leads us to believe that our softening of the problem formulation enables us to compute for large classes of problems where SDP may fail due to excessively high dimension. We note that this is only a conjecture and not proven in this paper.

The difficulty of high dimension also afflicts the Gröbner basis approach to the problem of polynomial positivity over a hypercube. For example, the Singular [19] software program quickly runs out of memory when trying to compute the Gröbner basis when the polynomial in question is the determinant of a 5×5 interval matrix, analogous to the example that follows in Section III-D. Using the softening of the problem formulation given in this paper, however, this problem can be addressed with modest computational resources.

E. Dilation Integrals

One of the main focal points of this paper is the so-called *dilation integral.* The simplest case is the sequence of α-functions

$$\epsilon_k(\alpha) \doteq \frac{1}{\mathbf{Vol}(\mathcal{X})} \int_{\mathcal{X}} (1 - \alpha f(x))^k dx$$

which are readily shown to be convex for $\alpha \geq 0$ and k being a positive even integer. The motivation for the use of such integrals is a basic inequality which we establish in Section II.

Namely, with

$$\mathcal{X}_{bad} \doteq \{x \in \mathcal{X} : f(x) \leq 0\}$$

the bound

$$\frac{\mathbf{Vol}(\mathcal{X}_{bad})}{\mathbf{Vol}(\mathcal{X})} \leq \epsilon_k(\alpha)$$

holds for all $\alpha \geq 0$ and all positive even integers k. As mentioned earlier, this type of volume bound can be used as a rationale for termination of computation. Moreover, it is established in the sequel that this bound is "sharp" in the sense that it converges to zero whenever the required polynomial positivity condition is satisfied.

F. Motzkin Polynomial

As further motivation, the example of Motzkin (see Fig. 1) is useful to demonstrate the power of our method mentioned above. Consider the polynomial

$$f(x) = 1 + x_1^2 x_2^2 \left(x_1^2 + x_2^2 - 3\right)$$

[1]The main theorems, presented in Section II of this paper, do hold in the more general case of $\mathcal{X}$ being a compact set. However, the simplicity of the computation of the dilation integral may be lost when the domain $\mathcal{X}$ is more complicated.

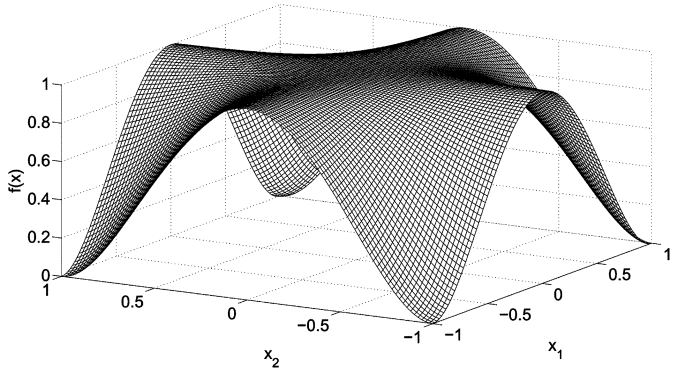


Fig. 1. Motzkin polynomial.

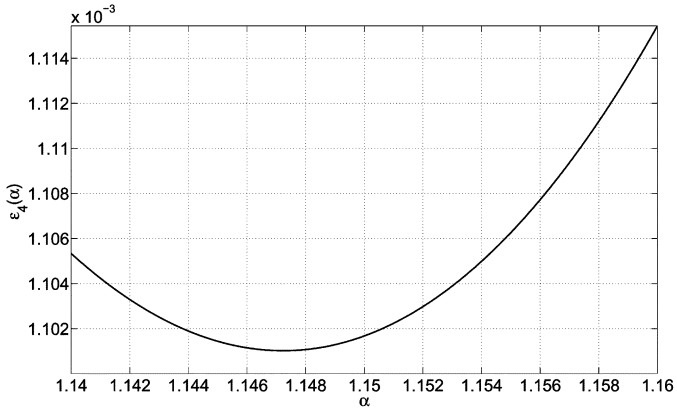


Fig. 2. Plot of $\epsilon_4(\alpha)$.

which is positive everywhere on the unit square except the vertices where it is zero. This polynomial does not admit an SOS representation. Moreover, as seen from the `mesh` plot above, for $\mathcal{X} = [-r; r]^2$ with $0 < r < 1$, the pair $(f, \mathcal{X})$ is positive.

To illustrate how the dilation integral method recovers this result, we take $r = 0.75$ and $k = 4$. A straightforward computation now leads to

$$\begin{aligned}\epsilon_4(\alpha) &= \frac{1}{(2 \cdot 0.75)^2} \int_{-0.75}^{0.75} \int_{-0.75}^{0.75} (1 - \alpha f(x))^4 dx_1 dx_2 \\ &\approx 0.7669\alpha^4 - 3.2272\alpha^3 + 5.1357\alpha^2 \\ &\quad - 3.6730\alpha + 1.\end{aligned}$$

Now, from the plot of this function in Fig. 2, we obtain the minimum value $\epsilon_4 \approx 0.001101$. In other words, we can guarantee that the positivity requirement is violated on a set whose volume is at most one tenth of one percent of the volume of $\mathcal{X}$. If a higher level of accuracy is required, a similar integration can be carried out to obtain $\epsilon_6 \approx 0.0001135$.

II. Dilation Integral Method: Simplest Form

In this section, we provide the simplest version of our results which serves as the "driver" for the dilation integral theory in this paper. That is, in this section, we provide results where there is only a scalar variable α to be optimized. In later sections, it is seen that one can obtain even tighter bounds by performing an optimization with α replaced by a vector of parameters which can be chosen by the user.

Recalling the discussion in Sections I-D and E, the main idea is that each of the integrals $\epsilon_k(\alpha)$ is a volume bound on the percentage violation. Once the integral is computed in closed form, a convex scalar minimization[2] with respect to $\alpha \geq 0$ results in

$$\epsilon_k \doteq \min_{\alpha \geq 0} \epsilon_k(\alpha)$$

which is also a volume bound. As seen in Theorem II-A, based on the main theorem in [1], the bounds obtained in this manner are *sharp* in the sense that $\mathbf{Vol}(\mathcal{X}_{bad}) = 0$ if and only if $\epsilon_k \to 0$ as $k \to \infty$.

A. Theorem (See Appendix for Proof)

For the given pair $(f, \mathcal{X})$ with associated dilation integrals

$$\epsilon_k(\alpha) = \frac{1}{\mathbf{Vol}(\mathcal{X})} \int_{\mathcal{X}} \Big(1 - \alpha f(x)\Big)^k dx$$

defined for positive even integers k, their minima

$$\epsilon_k \doteq \min_{\alpha \geq 0} \epsilon_k(\alpha)$$

are attained. Moreover, the following conditions hold:

1) For all k even, the percentage violation satisfies
$$\frac{\mathbf{Vol}(\mathcal{X}_{bad})}{\mathbf{Vol}(\mathcal{X})} \leq \epsilon_k.$$
2) If $(f, \mathcal{X})$ is a positive pair, then $\epsilon_k \to 0$.
3) If $\epsilon_k \to 0$, then $(f, \mathcal{X})$ is a non-negative pair with $\mathbf{Vol}(\mathcal{X}_{bad}) = 0$.

B. Interpretation of the Dilation Integral

In the proof of Theorem II-A, we chose a specific value of the parameter $\alpha > 0$ such that the inequality $-1 < 1 - \alpha f(x) < 1$ holds for all $x \in \mathcal{X}$ provided that the pair $(f, \mathcal{X})$ is positive. Subsequent exponentiation with increasing k reduces the magnitude of the integrand and therefore, arbitrarily small values of the integral are attainable. It is readily seen that the positivity of the pair $(f, \mathcal{X})$ is *equivalent* to the existence of a value of $\alpha > 0$ such that $|1 - \alpha f(x)| < 1$ for all $x \in \mathcal{X}$. In other words, by using the parameter α, the range of $f(\cdot)$ is scaled to the segment (0, 1).

This scaling appropriately corresponds to *dilation* of the range when $f(\mathcal{X})$ is "small," or to *contraction* when $f(\mathcal{X})$ is "large," and we refer to the integral $\epsilon_k(\alpha)$ as a *dilation integral*. Note that such a dilation takes place for any $0 < \alpha < 2/f_{\max}$.

C. Connection to Markov Inequality

Although we are working within a purely deterministic framework, it should be noted that the ϵ-softening approach which we pursue has a probabilistic analogue. That is, if we view x as a random vector uniformly distributed over $\mathcal{X}$, the quotient $\mathbf{Vol}(\mathcal{X}_{bad})/\mathbf{Vol}(\mathcal{X})$ represents the probability of violating the positivity specification. In this context, the basic

[2]In fact, the value of α which minimizes the function $\epsilon_k(\alpha)$ is the only real zero of the function $\epsilon_k'(\alpha) \doteq d/d\alpha \epsilon_k(\alpha)$.

relation $\mathbf{Vol}(\mathcal{X}_{bad})/\mathbf{Vol}(\mathcal{X}) \leq \epsilon_k(\alpha)$ can be thought of as the Markov inequality. Indeed

$$\begin{aligned}\frac{\mathbf{Vol}(\mathcal{X}_{bad})}{\mathbf{Vol}(\mathcal{X})} &= \mathsf{Prob}\{f(x) \leq 0\} \\ &= \mathsf{Prob}\{1-\alpha f(x) \geq 1\} \\ &\leq \mathsf{Prob}\{|1-\alpha f(x)| \geq 1\} \\ &\leq \mathsf{E}|1-\alpha f(x)|^k \\ &= \frac{1}{\mathbf{Vol}(\mathcal{X})}\int_{\mathcal{X}} |1-\alpha f(x)|^k dx = \epsilon_k(\alpha).\end{aligned}$$

Whereas in this probabilistic setting, the dilation integral bound $\epsilon_k(\alpha)$ above represents a *hard* theoretical bound on the probability of violation, we also point to a body of literature which can be used to obtain an empirical estimate of this probability via randomized algorithms. In contrast to this paper, such an estimate is certified in the sense of *probable approximate correctness*; e.g., see [20], [21] and references therein.

D. The Conditioner

The key idea proposed thus far is that a user can "exit" from a dilation integral computation if one attains a value of ϵ_k below some acceptable value; e.g., perhaps a 1% volume of violation is deemed appropriate. On the other hand, when a low ϵ_k is not attained, we claim that one might still be able to exit computation based on the following simple idea: For a given pair $(f, \mathcal{X})$, we define the *underlying conditioner* θ as the maximum percentage variation of $f(x)$ about the midpoint of its range. That is, with

$$f(\mathcal{X}) \doteq \{f(x) : x \in \mathcal{X}\} \doteq [f_{\min}, f_{\max}]$$

corresponding to minimization and maximization of $f(x)$, we obtain

$$\theta \doteq \frac{\sigma}{|f_0|}$$

where

$$\sigma \doteq \frac{1}{2}\Big(f_{\max} - f_{\min}\Big)$$

is the *spread* of $f(\mathcal{X})$ and

$$f_0 \doteq \frac{1}{2}\Big(f_{\max} + f_{\min}\Big)$$

is the *midpoint* of $f(\mathcal{X})$. Equivalently,

$$\theta = \frac{f_{\max} - f_{\min}}{|f_{\max} + f_{\min}|}.$$

To illustrate the ideas above, for the example $f(x) = [5, 25]$, we obtain $f_0 = 15$, $\sigma = 10$ and $\theta = 2/3$.

A couple of key observations are in order. First, while θ would typically be quite difficult to compute, it is seen in the sequel that one can generate a sequence θ_k which converges to θ. Second, for a pair $(f, \mathcal{X})$ failing to be positive, the conditioner θ may be arbitrarily large or possibly infinite. It is also simple to show, using basic geometry on the real line, that if there exists a point $x^0 \in \mathcal{X}$ such that $f(x^0) > 0$, then *the condition* $0 \leq \theta < 1$ is equivalent to positivity of the pair $(f, \mathcal{X})$. This suggests that as $\theta \to 1$, while positivity is guaranteed in a theoretical sense, in practice, one might violate positivity if there are small errors in the mathematical model for the pair $(f, \mathcal{X})$. Said another way, while theoretical positivity is guaranteed when $\theta \to 1$, the situation becomes increasingly "dangerous" from a practical point of view. That is, as θ gets close to unity, this indicates either that $\mathcal{X}_{bad}$ is non-empty or there exist values of $x \in \mathcal{X}$ which lead to a very "close call" regarding the desired positivity of $f(x)$. In such a case, while the pair $(f, \mathcal{X})$ might still be positive in a theoretical sense, we deem it to be non-positive in a *practical sense*.

E. Using θ to Exit Computation

Using the definition of ϵ_k, the chain of inequalities

$$\begin{aligned}\epsilon_k &= \min_{\alpha \geq 0} \epsilon_k(\alpha) \\ &\leq \epsilon_k\Big(\frac{1}{|f_0|}\Big) \\ &\leq \frac{1}{\mathbf{Vol}(\mathcal{X})}\int_{\mathcal{X}} \max_{x \in \mathcal{X}}\Big(1 - \frac{1}{|f_0|}f(x)\Big)^k dx \\ &= \max_{x \in \mathcal{X}}\Big(1 - \frac{1}{|f_0|}f(x)\Big)^k \\ &= \theta^k\end{aligned}$$

suggests that one can define an *estimator* for the conditioner θ. Namely, with

$$\theta_k \doteq \epsilon_k^{1/k}$$

we now have the lower bound

$$\theta \geq \theta_k$$

which can be used to provide an "exit point" for computations.

To illustrate, suppose dilation integral analysis leads to $\epsilon_{10} = 0.4$. Then, via the inequalities above, we obtain the result $\theta \geq (0.4)^{1/10} \approx 0.9124$. With this high level for the conditioner, in many applications, it is arguable that the pair $(f, \mathcal{X})$ should be deemed non-positive in a practical sense.

More generally, the relationship $\theta \geq \epsilon_k^{1/k}$, depicted in Fig. 3 for $2 \leq k \leq 18$, provides the basis for our "two-sided" exit theory for numerical computation. If we arrive at a large value of k and ϵ_k is still not sufficiently small, one can be certain that the conditioner θ is quite large; e.g., if ϵ_k remains above 0.4, by the time one reaches $k = 18$, we see from the plot that $\theta \geq 0.95$.

In summary, the key idea underlying the dilation integral method is that there are two scenarios which one can use to terminate numerical computation before k gets sufficiently large so as to render computation intractable. The first of these scenarios occurs if we arrive at the point where ϵ_k falls below some tolerable percentage volume of violation, call it ϵ_{tol}. In this case, we deem the pair $(f, \mathcal{X})$ to be *practically positive*. On the other hand, if ϵ_k fails to become acceptably small at a k level which is feasible for the required computation, a second opportunity to exit computation presents itself if θ_k gets suitably large; say $\theta > \theta_{tol}$, where θ_{tol} represents a level of conditioning above which we deem the pair to be non-positive in a practical sense. A formalization and generalization of the ideas above is given in the following theorem.

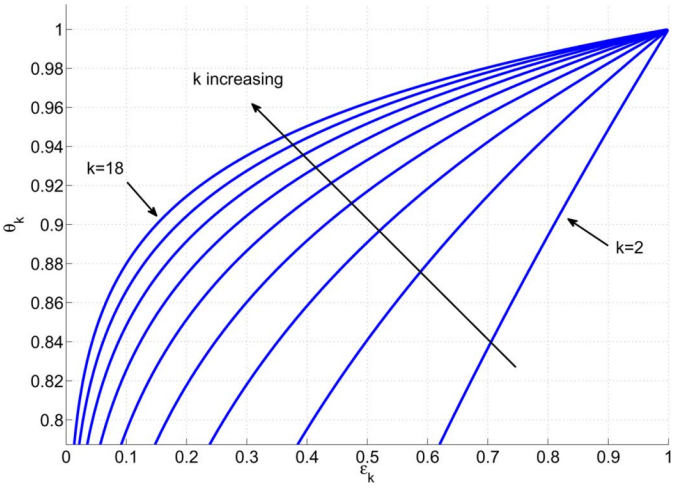


Fig. 3. Lower bounds on the conditioner θ.

F. Theorem (see Appendix for Proof)

If $(f, \mathcal{X})$ is a positive pair, it follows that:

$$\lim_{k\to\infty} \theta_k = \theta.$$

Alternatively, if $(f, \mathcal{X})$ is not a positive pair, then

$$\lim_{k\to\infty} \theta_k = 1.$$

Moreover, for both cases above, the sequence of estimates θ_k is non-decreasing.

G. Low k-Value Hypothesis

In this section, we provide some evidence in support of the hypothesis that practical positivity or non-positivity can often be certified with a value of k which may be much lower than that predicted by their theoretical bounds. To this end, we consider $k = 2$ and first note that the desired optimum for ϵ_2 and θ_2 can be provided in closed form. Assuming $(f, \mathcal{X})$ is a positive pair, a straightforward differentiation with respect to α leads to the closed form solution

$$\epsilon_2 = 1 - \frac{\left(\int_{\mathcal{X}} f(x)dx\right)^2}{\mathbf{Vol}(\mathcal{X}) \int_{\mathcal{X}} f^2(x)dx}.$$

Beginning with this formula, we now consider some benchmark cases. Our claim is that in some cases, with n suitably large, we can obtain a low value for ϵ_2 or a high value for θ_2 by exploiting the uncertainty structure. In other words, exit becomes possible with low computational effort. It is seen that increasing the dimension n of x results, on average, in a decrease in ϵ_2 and an increase in θ_2.

In order to make these claims more meaningful, we take $\mathcal{X}$ to be a hypercube with radius $r > 0$ centered at the origin. Now, to avoid trivialities, we assume $f(0) > 0$ and define $r_{\max}$ to be the maximal radius for $\mathcal{X}$ under which positivity is guaranteed. We now work with the quantity $\mu \doteq r/r_{\max}$, $0 \le \mu \le 1$ in order to gauge the behavior of $\epsilon_2 = \epsilon_2(\mu)$ with respect to μ, the percentage of the maximum allowable uncertainty.

For the linear function $f(x) = a^T x + \gamma$ with $\gamma > 0$, we introduce the squared-norm ratio $\rho_n(a) \doteq \|a\|_1^2/\|a\|_2^2$, where $\|a\|_1$ denotes the l_1-norm; i.e., $\|a\|_1 = \sum_{i=1}^n |a_i|$. Subsequently, after a calculation using $r_{\max} = \gamma/\|a\|_1$, we obtain

$$\epsilon_2(\mu) = \frac{\mu^2}{3\rho_n(a) + \mu^2}.$$

Identifying $a \in \mathbf{R}^n$ with different points in "problem space" and noting that $1 \le \rho_n(a) \le n$, for fixed n, it follows that:

$$\frac{\mu^2}{3n + \mu^2} \le \epsilon_2(\mu) \le \frac{\mu^2}{3 + \mu^2}.$$

These inequalities suggest that the "realized" volume estimate may decrease quite rapidly with the dimension n. Specifically, when n is large, since $\|a\|_1 \gg \|a\|_2$ in some "average" sense, one can expect $\epsilon_2(\mu)$ to be small. To test this hypothesis, we generated samples $a^{(i)}$ uniformly distributed over the surface of the euclidean norm unit ball in $\mathbf{R}^n$ and estimated the average behavior of both $\rho_n(a)$ and $\epsilon_2(\mu)$. Indeed, with $\hat{\rho}_n$ and $\hat{\epsilon}_2(\mu)$, respectively, denoting these estimates averaged over the set $\|a\|_2 = 1$

$$\hat{\rho}_n \approx 1 + \frac{2}{\pi}(n-1);$$
$$\hat{\epsilon}_2(\mu) \approx \frac{\mu^2}{\mu^2 + 1.91n + 1.09}$$

we see that as the dimension grows, on average, the use of $k = 2$ will be quite effective. To illustrate, for a positivity problem with $n = 40$ parameters and a radius of uncertainty which is 80% of $r_{\max}$, one can expect, on average, that the volume of violation $\mathbf{Vol}(\mathcal{X}_{bad})$ will be at most 1% of the total volume, $\mathbf{Vol}(\mathcal{X})$. It is important to note that the phenomenon above, improvement of average performance with increasing dimension n, is not attributable to the linearity of the problem under consideration. By carrying out more complicated algebraic calculations, rather similar results can be shown for other classes of functions. For example, for the multilinear function $f(x) = \gamma + \prod_{i=1}^n (1 - x_i)$ with $\gamma > 0$, a lengthy computation yields

$$\epsilon_2(\mu) = \frac{\left(1 - r(\mu) + \frac{1}{3}r^2(\mu)\right)^n - \left(1 - \frac{1}{2}r(\mu)\right)^{2n}}{\gamma^2 - 2\gamma\left(1 - \frac{1}{2}r(\mu)\right)^n - \left(1 - r(\mu) + \frac{1}{3}r^3(\mu)\right)^n}.$$

Using this formula, it can again be demonstrated that with n suitably large, practical positivity or non-positivity can often be certified with $k = 2$.

To conclude this section, it is important to emphasize that the arguments and results given above only *suggest* that low values of k will suffice. That is, the results given for the specialized linear and multilinear problem classes considered above provide no guarantee that the more general polynomial case will exhibit the same low-k behavior. In this regard, our point of view is that a necessary condition for the dilation integral approach to be efficacious is that it works well on simple benchmarks for which one can readily quantify the behavior of the method with respect to k. Finally, note that this issue of computational effort with respect to k serves to motivate much of the analysis in Section III.

III. Extension: Convex Dilation Integrals

The objective in this section is to improve the volume bound provided by the dilation integral method. This extension is motivated by the fact that we want ϵ_k converging as rapidly as possible in order to insure that the large number of terms in the symbolic integration does not lead to computational intractability. To this end, we now introduce additional parameters into the dilation integral which can be optimized to improve convergence of the volume bound ϵ_k. Indeed, for $a \in \mathbf{R}^\ell$, let $g(a,x)$ be a user-chosen real-valued function and define

$$f^a(x) \doteq g(a,x)f(x).$$

Now, with user-chosen compact restraint set $\mathcal{A}$, with $g(a,x) > 0$ for all $a \in \mathcal{A}$ and all $x \in \mathcal{X}$, it follows that the violation set for this new function f^a, defined as

$$\mathcal{X}^a_{bad} \doteq \{x \in \mathcal{X} : f^a(x) \leq 0\}$$

will be the same as the original set of violation; i.e.,

$$\mathcal{X}^a_{bad} = \mathcal{X}_{bad}.$$

Therefore, the dilation integral method, introduced in Section II, may now be used to obtain a sequence of estimates $\epsilon_k(\alpha,a)$ for the percentage violation of both the new function f^a and the original function f. Namely, for each $a \in \mathcal{A}$, the quantity

$$\epsilon_k(\alpha,a) \doteq \frac{1}{\mathbf{Vol}(\mathcal{X})}\int_{\mathcal{X}} (1-\alpha f^a(x))^k\, dx$$

is an upper bound on the percentage violation of both f^a and f, and the inequality

$$\frac{\mathbf{Vol}(\mathcal{X}_{bad})}{\mathbf{Vol}(\mathcal{X})} \leq \epsilon_k(\alpha,a)$$

holds for every $a \in \mathcal{A}$ and $\alpha \geq 0$. To achieve the tightest upper bound on the percentage violation for f, we seek

$$\epsilon^*_k \doteq \min_{\alpha\geq 0}\min_{a\in\mathcal{A}} \epsilon_k(\alpha,a).$$

A. Implications of Extra Parameters

With the extra degrees of freedom incorporated into the dilation integral, the main "payoff" is that with ϵ^*_k as above, we can accelerate the convergence of the volume bound by imposing a very simple condition on $g(a,x)$. Namely, if g is chosen to have the property that

$$g(a^0,x) = \text{constant}$$

for some $a^0 \in \mathcal{A}$, then with ϵ_k as defined in Section II, it is readily shown that

$$\frac{\mathbf{Vol}\,(\mathcal{X}_{bad})}{\mathbf{Vol}(\mathcal{X})} \leq \epsilon^*_k \leq \epsilon_k.$$

In other words, whenever the pair $(f,\mathcal{X})$ is positive, the inclusion of a-parameters will accelerate the rate of convergence of the volume estimates to zero. Even a simple scheme involving a finite set $\mathcal{A}$ will accelerate convergence, while requiring little extra computation.

Furthermore, if we select $\mathcal{A}$ to be a convex set, we claim that convex optimization methods can be applied to find ϵ^*_k in a computationally efficient manner, even for problems of high dimension; e.g., [22]. Note that no gridding of the entire parameter space is required under this condition; the estimate may be obtained by sweeping α and solving a convex program at each step. In the lemma below, we describe a simple scenario under which these desired properties on g and $\mathcal{A}$ are assured.

B. Lemma (see Appendix for Proof)

With

$$g(a,x) = 1 + a^T x$$

and

$$\mathcal{A} \doteq \{a \in \mathbf{R}^n : ||a||_1 \leq 1-\delta\}$$

for some positive $\delta < 1$, the function $\epsilon_k(\alpha,a)$ is convex in a for every fixed $\alpha \geq 0$.

C. Example (Robust Controllability)

We now revisit the motivating example from Section I-A to demonstrate the improvement seen when a function $g(a,x)$ is optimized in the estimate for the percentage violation. We consider the uncertain state-space system described in Section I-A

$$\begin{aligned} A(x) &= A_0 + x_2A_1 + x_1x_2A_2 + x_1x_2x_3A_3;\\ b(x) &= b_0 + x_1b_1 + x_2x_3b_2 + x_1x_2b_3 \end{aligned}$$

with randomly generated data

$$A_0 = \begin{pmatrix} 4.5015 & -0.1400 & -0.4355\\ -2.6885 & 3.9130 & -4.8150\\ 1.0685 & 2.6210 & 3.2140 \end{pmatrix};$$

$$A_1 = \begin{pmatrix} -0.1106 & 0.8436 & -0.1886\\ 0.2309 & 0.4764 & 0.8709\\ 0.5839 & -0.6475 & 0.8338 \end{pmatrix};$$

$$A_2 = \begin{pmatrix} -0.1795 & -0.2943 & -0.7222\\ 0.7873 & 0.6263 & -0.5945\\ -0.8842 & -0.9803 & -0.6026 \end{pmatrix};$$

$$A_3 = \begin{pmatrix} 0.2076 & -0.9695 & 0.8636\\ -0.4556 & 0.4936 & -0.0680\\ -0.6024 & -0.1098 & -0.1627 \end{pmatrix};$$

$$b_0 = \begin{pmatrix} 0.4357\\ 0.5894\\ -0.7466 \end{pmatrix};\quad b_1 = \begin{pmatrix} -0.0958\\ -0.8155\\ 0.4267 \end{pmatrix};$$

$$b_2 = \begin{pmatrix} -0.4863\\ 0.6714\\ -0.0440 \end{pmatrix};\quad b_3 = \begin{pmatrix} -0.6436\\ -0.7212\\ -0.2550 \end{pmatrix}$$

and parameter bounding set $\mathcal{X} = \{x : |x_i| \leq r\}$ where $r \geq 0$ is the variable uncertainty bound which we will consider.

To motivate the analysis to follow, we imagine an adaptive algorithm with parameter vector $x(t)$ evolving in the region $\mathcal{X}$.

TABLE I
PERCENTAGE VIOLATION ESTIMATES

r	ϵ_2	ϵ_2^*
0.05	0.0025	0.0018
0.10	0.0100	0.0019
0.15	0.0224	0.0045
0.20	0.0393	0.0087
0.25	0.0608	0.0147
0.30	0.0869	0.0232
0.35	0.1178	0.0348
0.40	0.1542	0.0503
0.45	0.1968	0.0705
0.50	0.2466	0.0961
1.00	0.8636	0.6823

With $\mathcal{C}(x)$ denoting the controllability matrix, it is often important to know that either the *loss of controllability manifold*

$$\mathcal{X}_{unc} = \{x \in \mathcal{X} : \det \mathcal{C}(x) = 0\}$$

is empty or that it is very unlikely that the parameter vector gets close to this set as time evolves. To complete this preliminary discussion, we note that a volume bound on the set where $\det \mathcal{C}(x) \leq 0$ also serves as a volume bound on $\mathcal{X}_{unc}$. Hence, if we obtain a bound which is very small, this is synonymous with a low likelihood that controllability will be lost as the parameter vector $x(t)$ roams over $\mathcal{X}$.

With this motivation in mind, we take $f(x) = \det \mathcal{C}(x)$. Noting that $f(0) \approx 20.0799$, we seek to determine the extent to which the pair $(f, \mathcal{X})$ is "practically" positive. Following Lemma III-B, we set $g(a, x) = 1 + a^T x$ and

$$\mathcal{A} = \left\{a \in \mathbf{R}^n : \|a\|_1 \leq 1 - 10^{-15}\right\}.$$

Since the requirement of nominal positivity is satisfied, we now obtain the percentage violation estimates ϵ_k and ϵ_k^* for various uncertainty radii from $r = 0.05$ to $r = 1$ which are shown in Table I. These values were generated for $k = 2$. The degree of improvement varies, but for most examples, the new volume estimate ϵ_k^* is one-third to one-fifth of the original estimate ϵ_k.

To complete the discussion of these numerical results, we raise the following question which will be addressed in the sequel: Are there ways to refine the computation of ϵ_k^* which makes it possible to handle much larger values of k in a computationally tractable manner? For example, if one deems the current volume bound at $r = 0.25$ to represent an unacceptably large risk, for cases when ϵ_k^* tends to zero, it is important to have computational capability to compute the dilation integral when k is large.

D. Example (Interval Matrix)

We now provide a second numerical example which illustrates the improvement in the estimate for percentage violation which may be seen when additional optimization parameters are introduced. Indeed, consider the basic case in which the function f is the determinant of a 3×3 interval matrix. That is

$$f(x) \doteq \begin{vmatrix} c_1x_1 + d_1 & c_2x_2 + d_2 & c_3x_3 + d_3 \\ c_4x_4 + d_4 & c_5x_5 + d_5 & c_6x_6 + d_6 \\ c_7x_7 + d_7 & c_8x_8 + d_8 & c_9x_9 + d_9 \end{vmatrix}$$

TABLE II
PERCENTAGE VIOLATION ESTIMATES

ϵ_2	ϵ_2^*
5.2×10^{-3}	6.3×10^{-5}
1.1×10^{-2}	1.2×10^{-8}
1.4×10^{-2}	5.9×10^{-7}
1.6×10^{-2}	4.1×10^{-5}
2.3×10^{-2}	8.4×10^{-5}
4.6×10^{-2}	3.6×10^{-5}
5.3×10^{-2}	3.3×10^{-8}
5.4×10^{-2}	6.7×10^{-9}
6.7×10^{-2}	1.3×10^{-5}
9.2×10^{-2}	4.2×10^{-9}

for real fixed parameters c_i and d_i. We now study positivity of $(f, \mathcal{X})$ by comparing the new estimate ϵ_k^* for the percentage violation with the old estimate ϵ_k. Again, following Lemma III-B, we set $g(a, x) = 1 + a^T x$ and

$$\mathcal{A} = \left\{a \in \mathbf{R}^n : \|a\|_1 \leq 1 - 10^{-15}\right\}.$$

Ten different interval matrix examples were generated by randomly selecting the parameters c_i and d_i. Each example was scaled so that the function f would be positive for all x in the unit cube. With the true percentage violation ϵ known to be zero, we computed ϵ_k and ϵ_k^* values for each example and compared the accuracy of these percentage violation estimates. Table II lists the values obtained for ϵ_k and ϵ_k^* for the 10 examples. The values were computed for $k = 2$.

As seen in Table II, the new method yields an upper bound ϵ_k^* on the percentage violation which is at least one order of magnitude smaller than ϵ_k. The degree of improvement that ϵ_k^* provides varies greatly from example to example, and does not seem to depend on conditioning.

E. Remarks

The ideas used to address the examples above can be readily adapted to solve a number of additional systems and control problems. To illustrate, we now sketch how our approach might be applied to a large class of robust stability problems with $n \times n$ state equation matrix $A(x)$ having entries which are multi-variable polynomials in the components x_i of x. While there are now a number of robust stability results in the literature which apply to systems having a so-called linear uncertainty structure, results for this nonlinear polynomial case have been slow in forthcoming. Going back to the eighties and nineties, we see that only special cases of this general problem class have been solved; e.g., see [23]–[25].

To sketch the main ideas noting that further work is needed to flesh out the details, we first observe that the associated Hurwitz matrix, call it $\mathcal{H}(x)$, also depends polynomially on the x_i. Letting $x = 0$ denote the nominal and assuming, without loss of generality, that $A(0)$ is stable with $\det \mathcal{H}(0) > 0$, we define the set of n polynomials $f_i(x) \doteq \det \mathcal{H}_i(x)$, where $\mathcal{H}_i(x)$ denotes the ith leading principle minor of the Hurwitz matrix. By exploiting continuous dependence on x, it can now readily be shown that the unstable subset of $\mathcal{X}$, call it $\mathcal{X}_{unst}$, is contained in the union of the sets

$$\mathcal{X}_{bad,i} \doteq \{x \in \mathcal{X} : f_i(x) \leq 0\}.$$

Therefore, by exploiting the inequality

$$\mathbf{Vol}(\mathcal{X}_{unst}) \leq \sum_{i=1}^{n} \mathbf{Vol}(\mathcal{X}_{bad,i})$$

we can prove the following: If we carry out n dilation integral tests and obtain sequences $\epsilon^*_{k,i}$ for the ith bad set above, robust stability is guaranteed if and only if each of these sequences tends to zero. Moreover, along the iterative path, we have the bound

$$\mathbf{Vol}(\mathcal{X}_{unst}) \leq \epsilon^*_k \doteq \sum_{i=1}^{n} \epsilon^*_{k,i}.$$

IV. Computational Aspects

One of the most attractive features of the dilation integral method is that it is extremely easy to implement. To illustrate this point, the MATLAB code generating the Motzkin plots in Fig. 2 is given below

```
r = 0.75,   k = 4
syms x y alpha
f = 1 + x^2 * y^2 * (x^2 + y^2 - 3)
h = (1 - alpha * f)^k
phi = int(int(h, x, -r, r), y, -r, r)
epsilon = (1/(4 * r^2)) * phi
ezplot(epsilon, 1.14, 1.16), grid.
```

As illustrated above, we use straightforward symbolic calculations as the main technical tool for computing the dilation integrals. While this method is computationally exact and easily performed, for example, using the Symbolic Math Toolbox in MATLAB, we note that computation may not be tractable for high dimensions of x and a large number of monomials in f. This is due to the possibility that high values of k will be required to obtain low volume certification for $\mathcal{X}_{bad}$ or high value of the conditioner θ. When k is large, the required symbolic expressions in the integrand defining ϵ_k can easily lead to an amount of intermediate data that may exceed MATLAB memory limitations. For a class of polynomial positivity problems, the approach described in [3] can be used to overcome this difficulty. We now describe the key elements of this approach.

A. Cubature Methods Avoiding Symbolic Computation

Beginning with the dilation integral, our objective is to show that one can obtain exact formulae for $\epsilon_k(\alpha)$ without recourse to symbolic computation. This is accomplished via evaluating the integrand numerically over a finite number of points using combinations of low-order quadrature formulae as described below. Such formulae largely reduce the computational burden and memory requirements. As a result, in many cases, dilation integrals can be efficiently computed for much higher values of exponent k anddimension n as compared to symbolic computation. Hence, this leads to an opportunity to improve the volume certification ϵ_k or the conditioning certification θ_k.

B. One-Dimensional Case (Quadrature Rules)

In this section, we briefly review the use of quadrature formulae for numerically computing integrals. Beginning with the one-dimensional case $n = 1$, where a well-known numerical computation method for approximating the definite integral

$$I[g] = \int_{-1}^{1} g(x)dx$$

of a continuous function g is given by so-called quadrature formulae; e.g., see [26]. An N-point quadratureformula with *nodes* ξ_m and *weights* w_m, $m = 1, \ldots, N$, is defined by

$$Q_N[g] \doteq \sum_{m=1}^{N} w_m g(\xi_m)$$

and we have

$$I[g] = Q_N[g] + R_N[g]$$

where $R_N[g]$ represents the error. Note that the computation above corresponds to evaluation of the function g over the N-point *grid* $\{\xi_1, \ldots, \xi_N\}$.

A quadrature formula is said to be *exact* for the function g if $R_N[g] = 0$. The *degree of exactness*, $\deg(Q_N)$, of a quadrature formula is defined as the maximum integer m such that the quadrature formula is exact for all polynomials of degree less than or equal to m, and there exists a polynomial p of degree $m + 1$ such that $R_N[p] \neq 0$. In particular, if the nodes ξ are chosen as the zeros of the Nth order Legendre orthogonal polynomial $P_N(x)$, and the weights are computed by integrating the associated Lagrange polynomials, then one obtains the maximum achievable degree of exactness

$$\deg(Q_N) = 2N - 1.$$

This is called a Gauss formula. To conclude this single-variable case, it is noted that if $f(x)$ is a polynomial of degree ν, with terms of degree up to $k\nu$, the application of a quadrature formula with degree of exactness $\deg(Q_N) = k\nu$ yields the exact numerical value for $\epsilon_k(\alpha)$.

C. Dilation Integrals Over Sparse Grids

The computational scheme outlined above can be extended to cover the multi-dimensional case $n > 1$. However, in its "naive" form, the quadrature formula requires the evaluation of the function over a multidimensional grid which is the Cartesian product of the corresponding one-dimensional grids for the components of x. Since this leads to a number of nodes which grows exponentially in the dimension n, the computations may become intractable as n increases.

To avoid this exponential growth, the ideas of Smolyak, originally proposed in [27], can be used. We consider linear combinations of *low-order quadrature formulae* which require evaluation of the function over a *sparse grid* , whose cardinality grows *polynomially* with respect to the dimension n. This approach has been subsequently studied in papers such as [28], [29], and exploited recently in [30] in the context of convex optimization.

By formally introducing an integer ℓ called *precision level*, the $N_{\ell,n}$-point *Smolyak formula* with nodes $\mathsf{x}_j \in \mathcal{X}$ and weights $\mathsf{w}_j \in \mathbf{R}$ may be written as

$$\mathcal{S}_{\ell,n}[g] = \sum_{j=1}^{N_{\ell,n}} \mathsf{w}_j g(\mathsf{x}_j).$$

See [27]–[30] and the references therein for precise formulations, calculation of the weights and nodes, properties and discussion on low-order cubature formulas and sparse grids. The first distinguished property of the Smolyak formula $\mathcal{S}_{\ell,n}$ is that it is exact for all multivariate polynomials in n variables of total degree[3] less than or equal to $2\ell+1$. That is, $\deg(\mathcal{S}_{\ell,n}) \geq 2\ell+1$. The second important property of $\mathcal{S}_{\ell,n}$ is that for fixed ℓ, the number of grid points grows polynomially in n as

$$N_{\ell,n} \approx \frac{2^\ell}{\ell!} n^\ell.$$

An important point to note is that, for given ℓ and n, the $N_{\ell,n}$ nodes and weights of the Smolyak formula can be computed *once for all* and stored for successive computations. Indeed, these universal quantities do not depend on the integrand, but only on ℓ and n. The procedure for actually computing the nodes and weights is the most time-consuming part of the proposed computational scheme. For this reason, a repository of nodes and weights for different values of ℓ, n has been created and is available from the authors upon request. Once these nodes and weights are obtained, the Smolyak formula above requires only ℓ, n evaluations of the integrand. Application of this approach to exact computation of dilation integrals is summarized in the theorem below.

D. Theorem (See Appendix for Proof)

With $f(x)$ being a multivariate polynomial of total degree ν, let $\ell = k\nu/2$, and let x_j and w_j, $j = 1, \ldots N_{\ell,n}$, be the nodes and weights of the (ℓ, n) Smolyak formula above. Define

$$\begin{aligned}
\mathsf{w} &\doteq [\mathsf{w}_1 \cdots \mathsf{w}_{N_{\ell,n}}]^T; \\
\mathsf{f} &\doteq [\mathsf{f}(\mathsf{x}_1) \cdots f(\mathsf{x}_{N_{\ell,n}})]^T; \\
\mathsf{f}^i &\doteq [f^i(\mathsf{x}_1) \cdots f^i(\mathsf{x}_{N_{\ell,n}})]^T; \\
\eta_i &\doteq \binom{k}{i} \mathsf{w}^T \mathsf{f}^i.
\end{aligned}$$

Then, for the dilation integral

$$\epsilon_k(\alpha) = \frac{1}{\mathbf{Vol}(\mathcal{X})} \int_{\mathcal{X}} (1 - \alpha f(x))^k \, dx$$

it follows that:

$$\epsilon_k(\alpha) = \frac{1}{\mathbf{Vol}(\mathcal{X})} \sum_{i=0}^{k} (-1)^i \eta_i \alpha^i.$$

E. Remark

The result of Theorem IV-D can be directly applied to the computation of the minimum of $\epsilon_k(\alpha)$. Indeed, recalling that the function $\epsilon_k(\alpha)$ is convex and attains the minimum at a finite unique point, the minimizer is directly computed as the unique real zero of the derivative. Under the conditions of Theorem IV-D, we obtain the unique minimizer

$$\alpha_k^* \doteq \arg\min_{\alpha \geq 0} \epsilon_k(\alpha)$$

as the only real zero of the polynomial

$$\epsilon_k'(\alpha) = \frac{1}{\mathbf{Vol}(\mathcal{X})} \sum_{i=1}^{k} (-1)^i i \eta_i \alpha^{i-1}.$$

F. Example (Resistive Ladder Network)

Consider the three-loop ladder network depicted in Fig. 4, which contains nine resistors. The *gain* of the network, defined as the ratio of the output and input voltages, is given by

$$g = \frac{V_{out}}{V_{in}}$$

and can be computed using mesh analysis. This leads to

$$g = \frac{R_2 R_5 R_8}{\det(M_R)}$$

where

$$M_R = \begin{bmatrix} R_1 + R_2 + R_3 & -R_2 & 0 \\ -R_2 & R_2 + R_4 + R_5 + R_6 & -R_5 \\ 0 & -R_5 & R_5 + R_7 + R_8 + R_9 \end{bmatrix}.$$

It is assumed that the values of the resistors R_i are uncertain, with nominal values $R_i^0 = 1$ and uncertain components x_i. Hence, for $i = 1, \ldots, 9$, we write

$$R_i(x) = R_i^0 + x_i; \quad |x_i| \leq r.$$

The robustness requirement in this illustrative analysis is that thegain $g(x)$ does not exceed a prespecified level for any value of x within the uncertainty bounding set $\mathcal{X} \doteq [-r, r]^9$. We formulate this requirement as follows: With g_0 denoting the gain obtained using the nominal resistors $R_i^0 = 1$, and $\gamma > 1$ denoting an acceptable *looseness* level, we insist that $g(x) < \gamma g_0$ must hold for all $x \in \mathcal{X}$. To recast this robustness requirement as a polynomial positivity problem, define

$$f(x) = \gamma g_0 \det(M_R(x)) - R_2(x) R_5(x) R_8(x).$$

To continue the calculation, suppose the looseness level is $\gamma = 1.5$ and the uncertainty radius is $r = 0.1975$. Since $f(x)$ is a multilinear function of x, the robustness requirement holds if and only if $f(x)$ is positive at each of the extreme points of $\mathcal{X}$. Thus, to benchmark our dilation integral computations to follow, we need only check the sign of $f(x)$ at the 512 extreme points of the uncertainty bounding set to verify that $(f, \mathcal{X})$ is not a positive pair at this looseness level[4]. Using MATLAB's symbolic engine to compute the dilation integrals for $k = 2$, we obtain $\epsilon_2 = 0.09133$, corresponding to $\theta_2 = 0.3022$, in approximately 20 seconds of computation time. For $k = 4$, MATLAB could not complete the computation on our computer due to memory constraints. Thus, for this example, it was not possible to come to a

[3]The total degree of a monomial $t(x) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ is equal to the sum of the exponents $\deg(t(x)) = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. The total degree of a multivariate polynomial is defined as the maximum total degree of its monomials.

[4]It is straightforward to see that the maximal gain for the network over all variations of x_i is attained by putting x_2, x_5, x_8 at their maximum admissible values and the rest of the uncertainties at their minimum values.

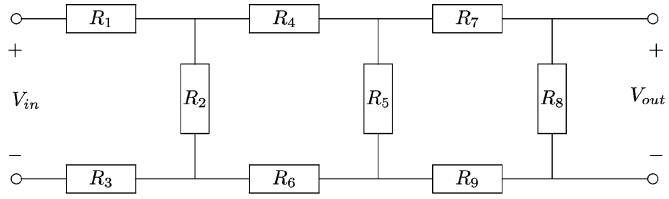


Fig. 4. Three-loop resistive ladder network.

TABLE III
PERCENTAGE VIOLATION ESTIMATES VIA SMOLYAK FORMULAS

k	$r = 0.25$	$r = 0.5$	$r = 1$
2	0.060813	0.24657	0.86357
4	0.0079143	0.1148	0.89823
6	0.0013289	0.064001	0.90900
8	0.00025498	0.039849	0.91377
10	5.3209e-005	0.027173	0.91645
12	1.1772e-005	0.020308	0.91817
14	2.7191e-006	0.016806	0.91936
16	6.4963e-007	0.015498	0.92024
18	4.7612e-007	0.015563	0.92091
20	3.2789e-007	0.019008	0.92144

conclusion of either practical positivity or non-positivity using straightforward symbolic computation of the dilation integrals.

However, using the method of computation associated with Theorem IV-D, the same value of ϵ_2 is obtained in approximately 0.1 seconds of computation time. Additionally, we are able to compute ϵ_k for higher values of k. Namely

$$\epsilon_4 = 0.026011; \quad \epsilon_6 = 0.012582; \quad \epsilon_8 = 0.008514.$$

With a percentage violation under one percent of the parameter space, in some situations, one may conclude that the robustness requirement is satisfied in a practical sense.

G. Example (Robust Controllability)

We once again revisit the robust controllability example from Sections I-A and III-C. Recall that the case $k = 2$ was previously studied and arguments were provided as to why computational tractability with higher k values would be important. To this end, Theorem IV-D is now applied to estimate the percentage violation. Using the Smolyak formulae and Theorem IV-D, we obtain estimates ϵ_k for k ranging 2 to 20, for three different values of the uncertainty radius r; see Table III. Note that in [1], the radius of robustness was also approximated by a brute force gridding with $N = 512{,}000$ samples evenly spaced over the parameter hypercube, yielding $r_{\max} \approx 0.4435$. Consistent with Theorem II-A, the sequence of estimates ϵ_k decreases for $r = 0.25$, as the pair $(f, \mathcal{X})$ is indeed a positive pair at this uncertainty radius. The cubature formulae allow the computation of ϵ_k for k as high as 20, leading to much tighter bounds on the percentage violation than would be available through straightforward symbolic computation; e.g., MATLAB's symbolic engine, which stalls for $k > 6$.

V. CONCLUSION

In this paper, a softened formulation has been given for the NP-hard problem of checking positivity of polynomials in several variables. This is accomplished by allowing for a small percentage volume of violation of the desired property. In particular, a numerically straightforward procedure, the dilation integral method, for computing a sequence of upper bounds for this quantity has been devised. Two major modifications of the basic scheme were also provided. The first one incorporates extra parameters in the integrand leading to improved convergence of the sequence of upper bounds while retaining convexity properties. Another modification exploits numerically exact evaluation of dilation integrals over sparse grids yielding considerable simplifications by exploiting structure in symbolic calculations.

Another important aspect of this paper is the relationship $\theta \geq \epsilon_k^{1/k}$. This inequality makes it possible to obtain a predetermined k-level which leads to a certification of either practical positivity or practical non-positivity. In this regard, it is of interest to note that it may be possible to improve the theory in this paper by defining the conditioner in a different manner. To illustrate one of the many possibilities, we introduce a tunable parameter $f_c > 0$ which is used for "centering" purposes and note that positivity of the pair $(f, \mathcal{X})$ is equivalent to the requirement that $f_c > |f(x) - f_c|$ for some $f_c > 0$ and all $x \in \mathcal{X}$. In fact, this extra degree of freedom provided by f_c might lead to improvements in computational efficiency. To see this, for each candidate f_c, let

$$g(x) \doteq f_c^2 - |f(x) - f_c|^2$$

and note the following: Use of the dilation integral method on the pair $(g, \mathcal{X})$ leads to a percentage volume bound, call it $\tilde{\epsilon}_k$ for $(g, \mathcal{X})$ which dominates the original volume bound ϵ_k for $(f, \mathcal{X})$ and also tends to zero for a positive pair with f_c appropriately tuned. This motivates defining a new conditioner

$$\tilde{\theta} \doteq \max_{x \in \mathcal{X}} |f(x) - f_c||f_c|$$

which can be justified in much the same way as the original θ; e.g., as $\tilde{\theta} \to 1$ with f_c optimally tuned, positivity of the pair $(f, \mathcal{X})$ is lost. For the special case when f_c happens to be the midpoint of the range $f(\mathcal{X})$, note that we obtain $\tilde{\theta} = \theta$. To exploit these ideas, we first compute the extrema for g. Namely

$$g_{\min} = f_c^2 - \tilde{\theta}^2 f_c^2; \quad g_{\max} = f_c^2$$

and thus, the conditioner for g, call it θ_g, is given by

$$\theta_g = \frac{\tilde{\theta}^2}{|2 - \tilde{\theta}^2|}.$$

Now using the fact that $\theta_g \geq \epsilon_k^{1/k}$, we obtain

$$\frac{\tilde{\theta}^2}{|2 - \tilde{\theta}^2|} \geq \epsilon_k^{1/k}$$

and a lengthy but straightforward computation leads to

$$\tilde{\theta} \geq \sqrt{2\epsilon_k^{1/k}(1 + \epsilon_k^{1/k})}.$$

Using universal bounds such as the one above, it may often be possible to terminate computations with low k-values. To illustrate, with $k = 8$ above, one can guarantee that $\tilde{\theta} \geq 0.9$ with $\epsilon_k \geq 0.05$.

Another area of interest for future research is the further refinement of the estimate ϵ_k^* for the percentage violation. As explained in Section III, this estimate involves the weighting func-

tion $g(a,x)$ designed to yield the smallest possible estimate for the percentage violation, subject to convexity and positivity conditions. It may be the case that certain classes of polynomials exhibit a particular structure that may be exploited to yield a choice of g which results in an improved estimate for the percentage violation. Along these lines, [4] and [31] provide results which accelerate the convergence of the conditioner estimate θ_k or multilinear positivity problems, such as the examples in Sections III-D and F. More specifically, with $f(x)$ being a multilinear function and taking $b_1, b_2, \ldots b_n$ to be tunable parameters, let

$$f^b(x) \doteq f(\text{sign}(x_1 - b_1), \text{sign}(x_2 - b_2), \ldots, \text{sign}(x_n - b_n))$$

and obtain the sequence of conditioner estimates

$$\theta_k^* \doteq \min_{\alpha \geq 0} \max_{b \in \mathcal{B}} \left(\frac{1}{\text{Vol}(\mathcal{X})} \int_{\mathcal{X}} \left(1 - \alpha f^b(x)\right)^k dx \right)^{1/k}$$

which can be shown to converge in a non-decreasing manner to θ and can also be used as a lower bound. The optimization of the parameters b_i within $\mathcal{B}$, a finite subset of $\mathcal{X}$, results in a much tighter lower bound on θ for a given exponent k. It is hoped that results like these could be developed for other classes of polynomial positivity problems.

Another strategy for improving volume estimates involves dividing the uncertainty bounding set $\mathcal{X}$ and examining the volume of violation over each individual subset. Preliminary results suggest that when $\mathcal{X}$ is divided in an adaptively strategic manner, the sum of the individual volume bounds is less than the volume bound obtained with one integration over all of $\mathcal{X}$.

These strategies focus on improving the percentage violation and conditioner estimates to achieve an acceptable result for low k; i.e., we wish to avoid the exponential increase of the number of dilation integrand terms as the index k increases. This combinatorial difficulty is not surprising, since the problem is just a reformulation of the originally NP-hard prototype. By way of further research, it would be interesting to analyze the accuracy of computations when using cubature formulae with degree of exactness less than the total degree of the integrand in order to evaluate dilation integrals for very high values of k, when necessary.

APPENDIX

Proof of Theorem II-A: The following proof is based on the proof of the main theorem in [1]. Noting that $\epsilon_k(\alpha)$ is a polynomial in α with highest order term $a_k \alpha^k$ with $a_k > 0$, it follows that $\epsilon_k(\alpha) \to \infty$ as $\alpha \to \infty$. Combining this fact with convexity of $\epsilon_k(\alpha)$, it must be the case that the minimum is attained. Now, to establish 1), it is noted that for arbitrary $\alpha \geq 0$ and k even, we have

$$\begin{aligned} \frac{\text{Vol}(\mathcal{X}_{bad})}{\text{Vol}(\mathcal{X})} &= \frac{1}{\text{Vol}(\mathcal{X})} \int_{f(x) \leq 0} dx \\ &\leq \frac{1}{\text{Vol}(\mathcal{X})} \int_{\mathcal{X}} \Big(1 - \alpha f(x)\Big)^k dx \\ &= \epsilon_k(\alpha). \end{aligned}$$

To establish 2), with $[f_{\min}, f_{\max}]$ corresponding to minimization and maximization of $f(x)$ over the domain $\mathcal{X}$, we define $f_0 \doteq 1/2\Big(f_{\max} + f_{\min}\Big)$ and consider the chain of inequalities

$$\begin{aligned} \epsilon_k &= \min_{\alpha \geq 0} \epsilon_k(\alpha) \leq \epsilon_k\Big(\frac{1}{|f_0|}\Big) \\ &\leq \frac{1}{\text{Vol}(\mathcal{X})} \int_{\mathcal{X}} \max_{x \in \mathcal{X}} \Big(1 - \frac{1}{|f_0|} f(x)\Big)^k dx \\ &= \max_{x \in \mathcal{X}} \Big(1 - \frac{1}{|f_0|} f(x)\Big)^k. \end{aligned}$$

The fact that $(f, \mathcal{X})$ is a positive pair now ensures that

$$0 \leq \max_{x \in \mathcal{X}} \left(1 - \frac{1}{|f_0|} f(x)\right) < 1$$

and it follows that $\epsilon_k \to 0$

To establish 3), it is now assumed that $\epsilon_k \to 0$. Then, in view of 1), we obtain

$$\frac{\text{Vol}(\mathcal{X}_{bad})}{\text{Vol}(\mathcal{X})} \leq \lim_{k \to \infty} \epsilon_k = 0.$$

To complete the proof, it remains to show that $\epsilon_k \to 0$ also implies that $(f, \mathcal{X})$ is a positive pair. Proceeding by contradiction, if $(f, \mathcal{X})$ is not a positive pair, then there exists some $x^0 \in \mathcal{X}$ with $f(x^0) < 0$. Hence, in view of the continuity of f, we have $f(x) < 0$ on some set $\mathcal{X}_0 \subseteq \mathcal{X}$ of positive volume. This implies that for all k even

$$\epsilon_k \geq \frac{\text{Vol}(\mathcal{X}_{bad})}{\text{Vol}(\mathcal{X})} \geq \frac{\text{Vol}(\mathcal{X}_0)}{\text{Vol}(\mathcal{X})} > 0.$$

This contradicts the standing hypothesis that $\epsilon_k \to 0$.

Proof of Theorem II-F: The following proof is based on the proof of the main theorem in [2]. Some basic properties of the functions $\theta_k(\alpha)$, which will be used later in the proof, are noted: First, the function $\theta_k(\alpha)$ is convex and corresponds to the L^k norm of $1 - \alpha f(x)$ with respect to the measure

$$\text{mes}(\mathcal{M}) \doteq \frac{\text{Vol}(\mathcal{M})}{\text{Vol}(\mathcal{X})}$$

on measurable subsets $\mathcal{M}$ of $\mathcal{X}$. Second, when $(f, \mathcal{X})$ is a positive pair, using the fact that $\theta_k(\alpha) \to \infty$ as $\alpha \to \infty$, it follows that every $\theta_k(\alpha)$ attains its minimum at a finite point $\alpha_k \geq 0$; hence, the quantities θ_k and α_k introduced above are well defined. Finally, note that $\theta_k(0) = 1$ guarantees that $\theta_k \leq 1$ is satisfied.

Now proceeding with the proof, we claim that the sequence of functions $\theta_k(\alpha)$ converges pointwise to the limit function

$$\theta(\alpha) \doteq \max_{x \in \mathcal{X}} |1 - \alpha f(x)|.$$

To prove this, let $\alpha \geq 0$ be fixed arbitrarily and define the continuous function

$$\varphi(x) \doteq \varphi_\alpha(x) \doteq |1 - \alpha f(x)|$$

on the compact set $\mathcal{X}$. Noting classical results for convergence of L^p norms, e.g., see [32], with the fact that $\left(\mathbf{Vol}(\mathcal{X})\right)^{1/k} \to 1$ as $k \to \infty$, we obtain

$$\lim_{k\to\infty} \left(\frac{1}{\mathbf{Vol}(\mathcal{X})}\int_{\mathcal{X}} \varphi^k(x)dx\right)^{1/k} = \max_{x\in\mathcal{X}} \varphi(x).$$

That is

$$\lim_{k\to\infty} \theta_k(\alpha) = \theta(\alpha)$$

and the claim of pointwise convergence is established. As the next step of the proof, we claim that the sequence of functions $\theta_k(\alpha)$ satisfies

$$\theta_k(\alpha) \le \theta_{k+2}(\alpha)$$

for all even k; i.e., it is non-decreasing. Indeed, for the non-negative continuous $\varphi(x)$ introduced above, by Hölder's inequality, see [32], it follows that:

$$\int_{\mathcal{X}} \varphi^k(x)dx \le \left(\mathbf{Vol}(\mathcal{X})\right)^{2/(k+2)} \left(\int_{\mathcal{X}} \varphi^{k+2}(x)dx\right)^{k/(k+2)}.$$

Now multiplying both sides by $\left(\mathbf{Vol}(\mathcal{X})\right)^{-1}$ and taking the kth root of both sides yields

$$\theta_k(\alpha) \le \left(\frac{1}{\mathbf{Vol}(\mathcal{X})}\int_{\mathcal{X}} \varphi^{k+2}(x)dx\right)^{1/(k+2)} = \theta_{k+2}(\alpha).$$

It then follows that the sequence $\theta_k = \min_\alpha \theta_k(\alpha)$ is non-decreasing. To continue the proof, we now claim that for every k, for the case when $(f, \mathcal{X})$ is a positive pair, the minimization of $\theta_k(\alpha)$ is attained with

$$0 \le \alpha \le \frac{2}{|f_{\min}|}.$$

That is, we have a fixed compact set $\mathcal{S}$ over which all minimizations can be restricted. To prove this, it suffices to show that with

$$\alpha > \frac{2}{|f_{\min}|}$$

we have

$$\theta_k(\alpha) > \theta_k(0).$$

In other words, such α exceeding this threshold cannot be optimal. To this end, it is first observed that for such α, we have $|\alpha f(x)| > 2$ for all $x \in \mathcal{X}$. Hence, for such α and all $x \in \mathcal{X}$, it follows that

$$|1 - \alpha\, f(x)| = \alpha|f(x)| - 1 > 1.$$

This implies that

$$\theta_k(\alpha) > 1 = \theta_k(0)$$

which establishes the claim. In view of this claim, we now have a non-decreasing pointwise convergent sequence of continuous functions $\theta_k(\alpha)$ over the compact set $\mathcal{S}$ with (continuous) limiting function $\theta(\alpha)$. By Dini's theorem, for example, see [33], it follows that the sequence of functions $\theta_k(\alpha)$ converges uniformly to $\theta(\alpha)$. Therefore, taking minimum with respect to α of both sides of the limiting relation $\lim_{k\to\infty} \theta_k(\alpha) = \theta(\alpha)$, we can change the order of the lim and min operations. This leads to

$$\min_\alpha \lim_{k\to\infty} \theta_k(\alpha) = \lim_{k\to\infty} \min_\alpha \theta_k(\alpha) = \lim_{k\to\infty} \theta_k$$

and arrive at

$$\lim_{k\to\infty} \theta_k = \min_\alpha \theta(\alpha).$$

The last step is to show that the right-hand side of the equality above is equal to θ if $(f, \mathcal{X})$ is a positive pair, and is equal to unity otherwise. This can be done by representing the limiting function $\theta(\alpha)$ in the "parametric" form

$$\begin{aligned}\theta(\alpha) &= \max_{f_{\min}\le\lambda\le f_{\max}} |1 - \alpha\lambda| \\ &= \max\{|1 - \alpha f_{\min}|, |1 - \alpha f_{\max}|\}.\end{aligned}$$

A straightforward analysis of the piecewise linear function $\theta(\alpha)$, performed under the assumption $f_{\max} > 0$, leads to a conclusion: If $f_{\min} > 0$, then

$$\min_\alpha \theta(\alpha) = \frac{f_{\max} - f_{\min}}{|f_{\max} + f_{\min}|} = \theta < 1$$

and if $f_{\min} \le 0$, then

$$\min_\alpha \theta(\alpha) = 1.$$

The proof of theorem is now complete.

Proof of Lemma III-B: The following proof is based on the proof of a theorem in [31]. With the dilation integrand

$$\Phi_\alpha(a, x) \doteq \left(1 - \alpha f(x)\left(1 + \sum_{i=1}^n a_i x_i\right)\right)$$

we see that for $k = 2$ and fixed $\alpha \ge 0$, the function $\Phi^2_\alpha(a, x)$ satisfies Jensen's inequality for every fixed $x \in \mathcal{X}$. Namely, for any $\lambda \in [0, 1]$ and any two elements a^0 and a^1 belonging to $\mathcal{A}$, straightforward algebraic manipulation shows that

$$\begin{aligned}&\lambda\Phi^2_\alpha(a^0, x) + (1 - \lambda)\Phi^2_\alpha(a^1, x) - \Phi^2_\alpha(\lambda a^0 + (1 - \lambda)a^1, x) \\ &\qquad = \alpha^2 f^2(x)\lambda(1 - \lambda)\left(\sum_{i=1}^n (a^0_i - a^1_i)x_i\right)^2.\end{aligned}$$

As this difference is clearly non-negative, Jensen's inequality is satisfied, and we conclude that $\Phi^2_\alpha(a, x)$ is convex in a for each fixed $x \in \mathcal{X}$. We may also conclude that for positive even integers k, the function $\Phi^k_\alpha(a, x)$ is convex in a for each $x \in \mathcal{X}$, being the result of the non-negative convex function $\Phi^2_\alpha(a, x)$ raised to a power greater than or equal to unity. Seeing that the dilation integrand $\Phi^k_\alpha(a, x)$ thus satisfies Jensen's inequality at

every fixed x in the domain of integration $\mathcal{X}$, we conclude that a Jensen-like inequality is also satisfied for the integral. Namely

$$\int_{x\in\mathcal{X}} \Phi_\alpha^k(\lambda a^0+(1-\lambda)a^1,x)dx \leq \lambda\int_{x\in\mathcal{X}}\Phi_\alpha^k(a^0,x)dx+(1-\lambda)\int_{x\in\mathcal{X}}\Phi_\alpha^k(a^1,x)dx.$$

Normalizing by $\mathbf{Vol}(\mathcal{X})$, we obtain

$$\lambda\epsilon_k(\alpha,a^0)+(1-\lambda)\epsilon_k(\alpha,a^1)\geq\epsilon_k(\alpha,\lambda a^0+(1-\lambda)a^1)$$

which completes the proof.

Proof of Theorem IV-D: First notice that the dilation integrand may be written as

$$\Big(1-\alpha f(x)\Big)^k=\sum_{i=0}^{k}(-1)^i\binom{k}{i}f^i(x)\alpha^i.$$

Then, since for the particular choice of ℓ, the Smolyak formula is exact for every n-polynomial of total degree less than or equal to $k\nu$, it follows that:

$$\int_{\mathcal{X}} f^i(x)\alpha^i\,dx=\sum_{j=1}^{N_{\ell,n}}\mathsf{w}_j f^i(\mathsf{x}_j)\alpha^i.$$

The statement then follows from algebraic manipulations and from the definition of w and f.

REFERENCES

[1] B. R. Barmish and P. S. Shcherbakov, "A dilation method for robustness problems with nonlinear parameter dependence," in *Proc. Amer. Control Conf.*, 2003, pp. 3834–3839.

[2] P. S. Shcherbakov and B. R. Barmish, "On the conditioning of robustness problems," in *Proc. Conf. Decision Control*, 2003, pp. 1932–1937.

[3] F. Dabbene and P. S. Shcherbakov, "Fast computation of dilation integrals for robustness analysis," in *Proc. Amer. Control Conf.*, 2007, pp. 2660–2665.

[4] B. R. Barmish and S. R. Ross, "New estimates for practical robust negativity of multilinear functions," in *Proc. Amer. Control Conf.*, 2007, pp. 5964–5969.

[5] N. Z. Shor, "Class of global minimum bounds of polynomial functions," *Cybernetics*, vol. 23, pp. 731–734, 1987.

[6] N. Z. Shor, "Quadratic optimization problems," *Soviet J. Comput. Syst. Sci.*, vol. 25, pp. 1–11, 1987.

[7] N. Z. Shor, *Nondifferentiable Optimization and Polynomial Problems*. Dordrecht, The Netherlands: Kluwer, 1998.

[8] G. Chesi *et al.*, "Solving quadratic distance problems: an LMI-based approach," *IEEE Trans. Automat. Control*, vol. 48, no. 2, pp. 200–212, Feb. 2003.

[9] P. Parrilo, "Semidefinite programming relaxations for semialgebraic problems," *Math. Programm. Series B*, vol. 96, pp. 293–320, 2003.

[10] J. Lasserre, "Global optimization with polynomials and the problem of moments," *SIAM J. Optim.*, vol. 11, pp. 796–817, 2001.

[11] *Positive Polynomials in Control*, D. Henrion and A. Garulli, Eds. Berlin, Germany: Springer, 2005.

[12] V. D. Blondel and J. N. Tsitsiklis, "A survey of computational complexity results in systems and control," *Automatica*, vol. 36, pp. 1249–1274, 2000.

[13] B. D. O. Anderson, N. K. Bose, and E. I. Jury, "Output feedback stabilization and related problems-solution via decision methods," *IEEE Trans. Automat. Control*, vol. AC-20, no. 1, pp. 53–66, Feb. 1975.

[14] P. Dorato, W. Wang, and C. T. Abdallah, "Application of quantifier elimination theory to robust multi-objective feedback design," *J. Symbolic Comput.*, vol. 11, pp. 1–5, 1996.

[15] J. Garloff and B. Graf, "Solving strict polynomial inequalities by Bernstein expansion," in *The Use of Symbolic Methods in Control System Analysis and Design*, N. Munro, Ed. London, U.K.: IEE, 1999, pp. 339–352.

[16] J. C. Doyle, "Analysis of feedback systems with structured uncertainties," *Proc. Inst. Elect. Eng. D*, vol. 129, pp. 242–250, 1982.

[17] E. Walach and E. Zeheb, "Sign test of multivariable real polynomials," *IEEE Trans. Circuits Syst.*, vol. CS-27, no. 7, pp. 619–625, Jul. 1980.

[18] Z. Lin, L. Xu, and N. K. Bose, "A tutorial on Gröbner bases with applications in signals and systems," *IEEE Trans. Circuits Syst. I*, vol. 55, no. 1, pp. 445–461, Feb. 2008.

[19] G.-M. Greuel, G. Pfister, and H. Schönemann, Singular 3.0 A Computer Algebra System for Polynomial Computations Centre Comput. Algebra, Univ. of Kaiserslautern, 2003 [Online]. Available: http://www.singular.uni-kl.de

[20] R. Tempo, G. C. Calafiore, and F. Dabbene, *Randomized Algorithms for Analysis and Control of Uncertain Systems*, ser. Communications and Control Engineering Series. London, U.K.: Springer-Verlag, 2005.

[21] M. Vidyasagar, *Learning and Generalization: With Applications to Neural Networks*, ser. Communications and Control Engineering Series, 2nd ed. London, U.K.: Springer-Verlag, 2002.

[22] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.

[23] J. Ackermann, *Robust Control: Systems with Uncertain Physical Parameters*. New York: Springer-Verlag, 1993.

[24] B. R. Barmish, *New Tools for Robustness of Linear Systems*. New York: Macmillan, 1994.

[25] S. P. Bhattacharyya, H. Chapellat, and L. H. Keel, *Robust Control: The Parametric Approach*. Upper Saddle River, NJ: Prentice Hall, 1995.

[26] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*. New York: Dover, 1970.

[27] S. A. Smolyak, "Quadrature and interpolation formulas for tensor products of certain classes of functions," *Doklady Akademii Nauk SSSR*, vol. 4, pp. 240–243, 1963.

[28] T. Gerstner and M. Griebel, "Numerical integration using sparse grids," *Numer. Algorithms*, vol. 18, pp. 209–232, 1998.

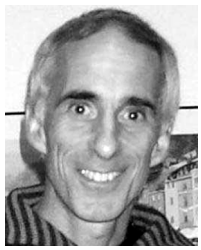
[29] K. Petras, "Smolyak cubature of given polynomial degree with few nodes for increasing dimension," *Numerische Matematik*, vol. 93, pp. 729–753, 2003.

[30] F. Dabbene, "Exact solution of uncertain convex optimization problems," in *Proc. Amer. Control Conf.*, 2007, pp. 2654–2659.

[31] S. R. Ross, "Contributions to the Theory of Robustness of Systems With Multilinear Uncertainty Structure," Ph.D. dissertation, Elect. Comp. Eng. Dept., Univ. of Wisconsin, Madison, 2006.

[32] W. Rudin, *Real and Complex Analysis*. New York: McGraw-Hill, 1974.

[33] T. M. Apostol, *Mathematical Analysis*. New York: Addison Wesley, 1973.



B. Ross Barmish (S'75–M'87–SM'93–F'95) received the B.S. degree from McGill University, Montreal, QC, Canada, in 1971 and the M.S. and Ph.D. degrees from Cornell University, Ithaca, NY, in 1972 and 1975, respectively, all in electrical engineering.

From 1975 to 1978, he served as Assistant Professor of Engineering and Applied Science at Yale University, New Haven, CT. From 1978 to 1984, he was as an Associate Professor of Electrical Engineering at the University of Rochester, Rochester, NY, and in 1984, he joined the University of Wisconsin, Madison, where he is currently Professor of Electrical and Computer Engineering. From 2001 to 2003, he was with the Department of Electrical Engineering and Computer Science, Case Western Reserve University, Cleveland, OH, where he served as Department Chair while holding the Nord Professorship. He has served as a consultant for a number of companies and is the author of the textbook *New Tools for Robustness of Linear Systems* (New York: Macmillan, 1994). Over the last year, his research has concentrated on the modelling and trading of markets for stocks and options.

Dr. Barmish received the Best Paper Award for Journal Publication on two consecutive occasions from the International Federation of Automatic Control. Over the years, he has been involved in a number of IEEE Control Systems Society activities such as associate editorships, conference chairmanships, and prize paper committees.

Pavel S. Shcherbakov was born in Moscow, Russia, in 1958. He received the M.S. degree in applied mathematics from the Department of Applied Mathematics, Moscow University of Transportation, in 1980, and the Candidate of Science degree (Russian equivalent of Ph.D.) in physics and mathematics and the Doctor of Science Degree from the Institute of Control Science, Russian Academy of Sciences, Moscow, in 1991 and 2004, respectively.

Since 1988, he has been with the Department of Robust and Adaptive Systems (Tsypkin Lab), Institute of Control Science, Moscow, where he is currently a Leading Researcher. His interests are in parametric robustness of control systems, probabilistic and randomized methods in control, linear matrix inequalities, and invariant ellipsoid methods in systems and control theory.

Sheila R. Ross (S'94–M'02) received the B.S. degree (with highest honors) in electrical engineering from Marquette University, Milwaukee, WI, in 1997, and the M.S. and Ph.D. degrees in electrical engineering from the University of Wisconsin-Madison, in 1998 and 2006, respectively.

Her current position is Adjunct Assistant Professor in the EECS department at the Milwaukee School of Engineering. She has previously worked as a lecturer in the EECS department at the University of California-Berkeley, and as a research intern at Los Alamos National Laboratory. Her research interests include robust control, optimization, and engineering education

Dr. Ross was a Finalist for Best Student Paper at the 2001 IEEE Conference on Decision and Control (CDC), received several fellowships including the NSF Graduate Research and NSF GK-12 Fellowships, and won two awards for her performance as a teaching assistant. Her work has been published in journals including IEEE TRANSACTIONS ON AUTOMATIC CONTROL, and she has presented a number of papers at the IEEE CDC and at the IEEE co-sponsored American Control Conference (ACC). She has also served as Reviewer for IEEE SIGNAL PROCESSING LETTERS and as a member of the Control Education Committee at the ACC.

Fabrizio Dabbene (M'01) received the Laurea degree in electrical engineering and the Ph.D. degree in systems and computer engineering in from the Politecnico di Torino, Italy, in 1995 and 1999, respectively.

He is currently holding a tenured research position at the institute IEIIT of the National Research Council (CNR) of Italy. He has held visiting and research positions at the University of Iowa, at the RAS Institute of Control Science, Moscow, and at Penn State University, University Park, PA. He is also involved in a collaboration with the Politecnico di Torino, where he teaches several courses in Systems and Control, and with Università degli Studi di Torino, where he conducts research on modeling of environmental systems. He is author or co-author of more than 70 publications, which include more than 20 articles published in international journals. He is co-author of the monograph *Randomized Algorithms for Analysis and Control of Uncertain Systems* (New York: Springer) and Editor of the book *Probabilistic and Randomized Methods for Design under Uncertainty* (New York: Springer). He is Associate Editor of *Automatica*. His research interests include robust control and identification of uncertain systems, randomized algorithms for systems and control, convex optimization and modeling of environmental systems.

Dr. Dabbene is an Associate Editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL. Since 2002, he has been a member of the Conference Editorial Board for the IEEE CSS. He is also chairing the Action Group on Probabilistic and Randomized Methods in Control (PRMC) of the IEEE Technical Committee on CACSD.