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Probabilistic design of LPV control systems☆

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Abstract

This paper presents an alternative approach to design of linear parameter-varying (LPV) control systems. In contrast to previous methods, which are focused on deterministic algorithms, this paper is based on a probabilistic setting. The proposed randomized algorithm provides a sequence of candidate solutions converging with probability one to a feasible solution in a finite number of steps. The main features of this approach are as follows: (i) The randomized algorithm gives a method for general LPV plants with state space matrices depending on scheduling parameters in a nonlinear manner. That is, the probabilistic setting does not need a gridding of the set of scheduling parameters or approximations such as a linear fractional transformation of the state space matrices. (ii) The proposed algorithm is sequential and, at each iteration, it does not require heavy computational effort such as solving simultaneously a large number of linear matrix inequalities.

Keywords: Randomized algorithms; Linear parameter-varying systems; Gain scheduling; Output feedback

1. Introduction

Linear parameter-varying (LPV) systems received a growing interest in the control community in recent years. In fact, these systems provide a good starting point for analysis and design of more general gain scheduling problems, see e.g., the survey papers of Leith and Leithead (2000), and Rugh and Shamma (2000). Since the original motivation for introducing gain scheduling is to cope with various plant nonlinearities, a critical issue of the resulting LPV model is that it generally depends on the scheduling parameters in a nonlinear fashion. As a result, these problems are often shown to be equivalent to the solution of some parameter-dependent linear matrix inequalities (LMIs). However, these inequalities are still nonlinear with respect to the scheduling parameters, and therefore it is generally impossible to solve them exactly.

In order to address this critical issue, two different approaches are generally followed in the LPV literature. The first approach, denoted as "approximation," amounts to restricting the attention to a specific class of functions of the scheduling parameters. For example, one can assume that the matrices of the LPV model are multi-affine functions or linear fractional transformations of the underlying parameters, see for instance, Becker and Packard (1994), Apkarian and Gahinet (1995), and Köse, Jabbari, and Schmitendorf (1998). The original problem is then reduced to more tractable formulae which involve a finite number of LMIs but, unfortunately, some conservatism is introduced in the approximation.

The second approach, often denoted as "gridding," is to grid the bounding set of parameters, see, e.g., Becker and Packard (1994), Apkarian and Adams (1998), and Wu and Grigoriadis (2001). In this case, the original problem is reformulated by means of the solution of a finite number of LMIs, but this number depends on the grid points and therefore may dramatically increase with the number of scheduling parameters. Moreover, the fulfillment of the LMIs at the grid points does not give any guarantee that they

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are also satisfied for the whole parameter set. Obviously, other approaches which combine both methods may be also used, depending on the practical problem which should be handled. However, the core of the problem is solving simultaneously a fairly large number of LMIs, and therefore computational complexity becomes a critical issue, as discussed in Toker (1998).

The objective of this paper is to develop an alternative method to handle LPV systems. In this method, very mild assumptions regarding functional approximations of the scheduling parameters are made. In addition, no gridding technique of the parameter set is required. Instead, the original problem is formulated in a probabilistic setting, see, for instance, the papers of Stengel and Ray (1991), Khargonekar and Tikku (1996), Barmish and Lagoa (1997), Tempo, Bai, and Dabbene (1997), Tempo and Dabbene (2001), and Vidyasagar (2001). The scheduling parameters are treated as random variables so that a randomized algorithm is proposed. In this setting, we first need to generate random vectors within the parameter set, see, e.g., Calafiore, Dabbene, and Tempo (2000) for polynomial-time algorithms. Secondly, we introduce a gradient-based algorithm which computes solutions of LMIs at each step. Indeed, a nice feature of this algorithm is that it is sequential and, at each step of the sequence, it generates a random vector sample corresponding to the scheduling parameters and executes a subgradient-based minimization defined by the LPV control problem. Hence, each iteration does not require heavy computational effort such as solving simultaneously a large number of LMIs. Obviously, in contrast to classical approaches for handling LPV systems, only a probabilistic solution is given, and a certain risk-level should be accepted by the user. The main result of the paper is to prove that, under relatively mild assumptions, this randomized algorithm provides a feasible solution with probability one in a finite number of steps.

We observe that the randomized algorithm proposed in this paper is an extension of the work of Polyak and Tempo (2001), where probabilistic robust design of LQ regulators is studied for systems affected by nonlinear parametric uncertainty. The key difference with the previous paper is therefore to deal with output feedback problems instead of state feedback. In fact, in the LPV case the solvability conditions are characterized by some structured and coupled matrix inequalities while in the LQR case they are simply given by one unstructured quadratic matrix inequality (QMI). Hence, one of the novelties of this paper is to suitably modify the stochastic gradient algorithm previously given in order to handle the specific structure induced by the LPV problem. Furthermore, we give a comprehensive result on the convergence of this type of randomized algorithms, from the viewpoint of expected number of iterations, boundedness of the solutions, and monotone-like convergence. These convergence results form a part of the contribution of the paper. Although in this paper we utilize a specific structure of LPV control problems, we notice that there are a few recent-related results in the general context of parameter-dependent LMIs. For example, an approach for approximate LMI feasibility is considered in Barmish and Shcherbakov (2002) and a specific randomized algorithm is given in Calafiore and Polyak (2001) for general LMIs. A detailed analysis of computational complexity of a similar randomized algorithm, with an additional control parameter, is derived in Oishi and Kimura (2001).

The paper is organized as follows. In Section 2, we state the problem formulation and we recall the solution for the LPV control problem proposed by Becker and Packard (1994). In particular, for the sake of readability, we discuss a simplified version of the problem, considering the case when some regularity assumptions hold and the rate variation of the parameters is unbounded. The general case of bounded rate change and no regularity assumptions is thoroughly discussed in Appendix A. Then, in Section 3, we present the main result of this paper. That is, we introduce a randomized algorithm to solve the LPV control problem and prove its probabilistic convergence. In Section 4, we demonstrate the effectiveness of the proposed algorithm for two examples studying, respectively, the lateral motion of an aircraft and a two-link flexible manipulator. Finally, in Section 5, we give concluding remarks and make some comments on possible extensions of these results for nonlinear constrained systems.

In this paper, we use the following notation. For $x \in \mathbb{R}$, $x > 0$, we denote by $\lceil x \rceil$ the minimum integer greater than or equal to x. Let $\mathbb{R}^{n\times m}$ be a Euclidean space equipped with Frobenius norm $\|A\| = (\operatorname{tr} A^{\mathrm{T}} A)^{1/2}$ and inner product $\langle A, B\rangle = \operatorname{tr} A^{\mathrm{T}} B$, where A^{T} denotes the transpose of A. The set of symmetric matrices $A \in \mathbb{R}^{n\times n}$ is a subspace of this Euclidean space. For any symmetric matrix $A \in \mathbb{R}^{n\times n}$, we define the projection $[A]^+$ on the cone of positive semidefinite matrices as $[A]^+ \doteq \arg\min_{X\in\mathscr{C}} \|A - X\|$, where $\mathscr{C} \doteq \{X \in \mathbb{R}^{n\times n}\colon X = X^{\mathrm{T}} \geqslant 0\}$. This projection can be computed explicitly by means of eigenvalue-eigenvector decomposition (Polyak, 1964) or the Schur decomposition. That is, for $A = A^{\mathrm{T}}$, we have

$$[A]^+ = \begin{cases} 0 & \text{if } A \leqslant 0, \\ U_p \Lambda_p U_p^{\mathrm{T}} & \text{otherwise,} \end{cases}$$

where the matrices U_p and Λ_p are taken from the real Schur decomposition of A (Grigoriadis & Skelton, 1996; Zhu, Rotea, & Skelton, 1997; Skelton, Iwasaki, & Grigoriadis, 1998). That is,

$$A = [\, U_p \quad U_n \,] \begin{bmatrix} \Lambda_p & 0 \\ 0 & \Lambda_n \end{bmatrix} \begin{bmatrix} U_p^{\mathrm{T}} \\ U_n^{\mathrm{T}} \end{bmatrix},$$

where Λ_p and Λ_n are diagonal matrices containing, respectively, the nonnegative and the strictly negative eigenvalues of A, and $[\, U_p \;\; U_n \,]$ is an orthogonal matrix.

2. Quadratic LPV L_2 control problem

In this section, we present some standard material on the LPV control problem. Additional details may be found, for example, in Becker and Packard (1994), Asai and Hara (1999), Rugh and Shamma (2000), and Scherer (2001). In particular, we focus on the problem formulation for the special case discussed in Becker and Packard (1994), where some regularity assumptions hold. The reader interested in a general formulation of the LPV problem may refer to Appendix A, where the equations and a restatement of the results are given.

Let us consider an LPV plant of the form

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(\theta(t)) & B_1(\theta(t)) & B_2(\theta(t)) \\ C_1(\theta(t)) & D_{11}(\theta(t)) & D_{12}(\theta(t)) \\ C_2(\theta(t)) & D_{21}(\theta(t)) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \\ u(t) \end{bmatrix}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $d(t) \in \mathbb{R}^{n_d}$ is the disturbance, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $e(t) \in \mathbb{R}^{n_e}$ is the controlled output, and $y(t) \in \mathbb{R}^{n_y}$ is the measurement output. For all $t \geqslant 0$, we assume that $\theta(t) \in \Theta$, where Θ is a subset of $\mathbb{R}^{n_s}$. According to standard LPV literature, the scheduling parameter $\theta(t)$ is not known in advance but can be measured on-line.

The matrices $A(\theta)$, $B_1(\theta)$, $B_2(\theta)$, $C_1(\theta)$, $C_2(\theta)$, $D_{11}(\theta)$, $D_{12}(\theta)$, and $D_{21}(\theta)$ are continuous functions of θ bounded on $\theta \in \Theta$. However, it should be noted that they may be *nonlinear* functions of θ.

We now introduce regularity assumptions needed to simplify the resulting equations.

Regularity assumptions:

(a) There is no direct term, i.e., $D_{11}(\theta) \equiv 0$.
(b) The orthogonality conditions

$$D_{12}^{\mathrm{T}}(\theta)[\, C_1(\theta) \quad D_{12}(\theta)\,] = [\,0 \quad I\,],$$

$$\begin{bmatrix} B_1(\theta) \\ D_{21}(\theta) \end{bmatrix} D_{21}^{\mathrm{T}}(\theta) = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{2}$$

are satisfied for all $\theta \in \Theta$.
(c) The vector $\theta(t)$ is a piecewise continuous function of t with a finite number of discontinuities in any interval.

For plant (1), we adopt an LPV full-order strictly proper controller of the form

$$\begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_c(\theta(t)) & B_c(\theta(t)) \\ C_c(\theta(t)) & 0 \end{bmatrix} \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}, \tag{3}$$

where $x_c(t) \in \mathbb{R}^n$ is the state. The matrices $A_c(\theta)$, $B_c(\theta)$ and $C_c(\theta)$ are continuous functions of θ, bounded on $\theta \in \Theta$.

Then, we write the closed-loop system

$$\begin{bmatrix} \dot{x}_{cl}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A_{cl}(\theta(t)) & B_{cl}(\theta(t)) \\ C_{cl}(\theta(t)) & 0 \end{bmatrix} \begin{bmatrix} x_{cl}(t) \\ d(t) \end{bmatrix}, \tag{4}$$

where

$$x_{cl}(t) = \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix},$$

$$A_{cl}(\theta) = \begin{bmatrix} A(\theta) & B_2(\theta)C_c(\theta) \\ B_c(\theta)C_2(\theta) & A_c(\theta) \end{bmatrix},$$

$$B_{cl}(\theta) = \begin{bmatrix} B_1(\theta) \\ B_c(\theta)D_{21}(\theta) \end{bmatrix},$$

$$C_{cl}(\theta) = \left[\, C_1(\theta) \quad D_{12}(\theta)C_c(\theta) \,\right].$$

Problem 1 (Quadratic LPV L_2 control). Suppose that the Regularity Assumptions are satisfied. Given $\gamma > 0$, find matrices $A_c(\theta)$, $B_c(\theta)$, and $C_c(\theta)$ such that there exist $X_{cl} = X_{cl}^{\mathrm{T}} \in \mathbb{R}^{2n\times 2n}$ and $\varepsilon > 0$ satisfying

$$A_{cl}(\theta)X_{cl} + X_{cl}A_{cl}^{\mathrm{T}}(\theta) + X_{cl}C_{cl}^{\mathrm{T}}(\theta)C_{cl}(\theta)X_{cl} + \gamma^{-2}B_{cl}(\theta)B_{cl}^{\mathrm{T}}(\theta) + \varepsilon I \leqslant 0, \tag{5}$$

$$X_{cl} \geqslant 0,$$

for all $\theta \in \Theta$.

We notice that inequality (5) implies that the closed-loop system is quadratically stable and its L_2 norm satisfies

$$\sup_{d \in L_2/\{0\}} \frac{(\int_0^\infty e^{\mathrm{T}}(t)e(t)\,\mathrm{d}t)^{1/2}}{(\int_0^\infty d^{\mathrm{T}}(t)d(t)\,\mathrm{d}t)^{1/2}} < \gamma \tag{6}$$

for all $\theta(t) \in \Theta$ and $x_{cl}(0) = 0$.

Next, we state a lemma which shows that the condition of the quadratic LPV L_2 control problem is equivalent to the solution of coupled matrix inequalities. This lemma is stated in a slightly different form than in Becker and Packard (1994). That is, in the lemma below there are two QMIs and one LMI with two symmetric matrix variables, while in the original statement three LMIs with two symmetric positive definite matrix variables are involved.

Lemma 2. *The quadratic LPV L_2 control problem is solvable if and only if there exist $X = X^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $Y = Y^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, and $\varepsilon > 0$ satisfying*

$$P(X,\theta) \doteq A(\theta)X + XA^{\mathrm{T}}(\theta) + XC_1^{\mathrm{T}}(\theta)C_1(\theta)X + \gamma^{-2}B_1(\theta)B_1^{\mathrm{T}}(\theta) - B_2(\theta)B_2^{\mathrm{T}}(\theta) + \varepsilon I \leqslant 0, \tag{7}$$

$$Q(Y,\theta) \doteq A^{\mathrm{T}}(\theta)Y + YA(\theta) + YB_1(\theta)B_1^{\mathrm{T}}(\theta)Y + \gamma^{-2}C_1^{\mathrm{T}}(\theta)C_1(\theta) - C_2^{\mathrm{T}}(\theta)C_2(\theta) + \varepsilon I \leqslant 0, \quad (8)$$

$$R(X,Y) \doteq -\begin{bmatrix} X & \gamma^{-1}I \\ \gamma^{-1}I & Y \end{bmatrix} \leqslant 0 \quad (9)$$

for all $\theta \in \Theta$. Furthermore, if the above conditions hold, then there exist another set of $X = X^{\mathrm{T}} > 0$, $Y = Y^{\mathrm{T}} > 0$, and $\varepsilon > 0$ satisfying (7), (8) and $R(X,Y) < 0$ which gives the matrices of an LPV controller

$$A_c(\theta) = A(\theta) - Y^{-1}C_2^{\mathrm{T}}(\theta)C_2(\theta) - B_2(\theta)B_2^{\mathrm{T}}(\theta)(X - \gamma^{-2}Y^{-1})^{-1} + \gamma^{-2}Y^{-1}C_1^{\mathrm{T}}(\theta)C_1(\theta) + \gamma^{-2}Y^{-1}(Q(Y,\theta) - \varepsilon I)(XY - \gamma^{-2}I)^{-1},$$

$$B_c(\theta) = Y^{-1}C_2^{\mathrm{T}}(\theta),$$

$$C_c(\theta) = -B_2^{\mathrm{T}}(\theta)(X - \gamma^{-2}Y^{-1})^{-1}. \quad (10)$$

We observe that solving the matrix equations (7)–(9) is a quite difficult task. This is a consequence of the fact that we need to check these conditions *for all* values of the scheduling parameters $\theta \in \Theta$ because the coefficient matrices may be nonlinear functions of θ.

3. Randomized algorithm for probabilistic design

In this section, we propose a probabilistic solution for the quadratic LPV L_2 control problem. The key idea in this method is to assume that θ is a random vector with given probability density function (pdf) $f_\theta(\theta)$. The proposed randomized algorithm is sequential. At each step of the sequence, first a random vector sample θ^k is generated and secondly a subgradient-based minimization is executed for obtaining a feasible solution $(X,\ Y)$ of the inequalities given in Lemma 2.

In the next subsection, we reformulate the original feasibility problem defined in Lemma 2 as an optimization problem. Then, in Section 3.2, we present a randomized algorithm to solve this optimization problem and we state probabilistic convergence results.

3.1. Objective function and subgradients

To recast conditions (7)–(9) as a minimization problem, we introduce a matrix-valued function of the form

$$V(X,Y,\theta) \doteq \begin{bmatrix} P(X,\theta) & 0 & 0 \\ 0 & Q(Y,\theta) & 0 \\ 0 & 0 & R(X,Y) \end{bmatrix}. \quad (11)$$

Then, we define a scalar function that plays the role of objective function in the algorithm presented in the next subsection

$$v(X,Y,\theta) \doteq \|[V(X,Y,\theta)]^+\| = (\|[P(X,\theta)]^+\|^2 + \|[Q(Y,\theta)]^+\|^2 + \|[R(X,Y)]^+\|^2)^{1/2}. \quad (12)$$

The scalar-valued function $\|[\,\cdot\,]^+\|$ does not define a norm on the space of X, Y, θ. However, we note that $v(X,Y,\theta) \geqslant 0$ holds for all $(X,\ Y)$ and $\theta \in \Theta$. We also observe that the feasibility set is characterized as the set of $(X,\ Y)$ such that $v(X,Y,\theta)$ takes the minimum value zero for all $\theta \in \Theta$. Thus, the feasibility problem of finding a solution (X,Y) of the matrix inequalities (7)–(9) is recast as the problem of minimizing the scalar function $\sup\{v(X,Y,\theta)\colon\ \theta \in \Theta\}$. This minimization is performed by means of the randomized algorithm proposed in the next subsection.

The next lemma is a technical tool which shows that the subgradients of $v(X,Y,\theta)$ can be computed explicitly. These subgradients are subsequently used in the implementation of the randomized algorithm.

Lemma 3. *The function $v(X,Y,\theta)$ is convex in $(X,\ Y)$ and its subgradients are given by*

$$\partial_X\{v(X,Y,\theta)\} = \frac{[P(X,\theta)]^+}{v(X,Y,\theta)}(A(\theta) + XC_1^{\mathrm{T}}(\theta)C_1(\theta)) + (A^{\mathrm{T}}(\theta) + C_1^{\mathrm{T}}(\theta)C_1(\theta)X)\frac{[P(X,\theta)]^+}{v(X,Y,\theta)} - [\,I \quad 0\,]\frac{[R(X,Y)]^+}{v(X,Y,\theta)}\begin{bmatrix} I \\ 0 \end{bmatrix},$$

$$\partial_Y\{v(X,Y,\theta)\} = \frac{[Q(Y,\theta)]^+}{v(X,Y,\theta)}(A^{\mathrm{T}}(\theta) + YB_1(\theta)B_1^{\mathrm{T}}(\theta)) + (A(\theta) + B_1(\theta)B_1^{\mathrm{T}}(\theta)Y)\frac{[Q(Y,\theta)]^+}{v(X,Y,\theta)} - [\,0 \quad I\,]\frac{[R(X,Y)]^+}{v(X,Y,\theta)}\begin{bmatrix} 0 \\ I \end{bmatrix}$$

if $v(X,Y,\theta) > 0$ or

$$\partial_X\{v(X,Y,\theta)\} = 0, \qquad \partial_Y\{v(X,Y,\theta)\} = 0$$

if $v(X,Y,\theta) = 0$.

Proof. First, we observe that the convexity of $v(X,Y,\theta)$ in (X,Y) is obvious because the quadratic terms are positive semidefinite. Secondly, we notice that the subgradients are zero whenever $[V(X,Y,\theta)]^+ = 0$. Thus, in the remaining of the proof we concentrate our attention on the case of $[V(X,Y,\theta)]^+ \neq 0$.

To this end, we observe that

$$V(X + X_\Delta, Y + Y_\Delta, \theta) = V(X, Y, \theta) + V_\Delta,$$

where

$$V_\Delta = \begin{bmatrix} P_\Delta & 0 & 0 \\ 0 & Q_\Delta & 0 \\ 0 & 0 & R_\Delta \end{bmatrix},$$

$$P_\Delta = (A(\theta) + XC_1^{\mathrm{T}}(\theta)C_1(\theta))X_\Delta + X_\Delta(A^{\mathrm{T}}(\theta) + C_1^{\mathrm{T}}(\theta)C_1(\theta)X),$$

$$Q_\Delta = (A^{\mathrm{T}}(\theta) + YB_1(\theta)B_1^{\mathrm{T}}(\theta))Y_\Delta + Y_\Delta(A(\theta) + B_1(\theta)B_1^{\mathrm{T}}(\theta)Y),$$

$$R_\Delta = -\begin{bmatrix} X_\Delta & 0 \\ 0 & Y_\Delta \end{bmatrix} = -\begin{bmatrix} I \\ 0 \end{bmatrix} X_\Delta [\, I \quad 0 \,] - \begin{bmatrix} 0 \\ I \end{bmatrix} Y_\Delta [\, 0 \quad I \,].$$

Now, we observe that $v(X, Y, \theta)$ is differentiable and thus we obtain

$$v(X + X_\Delta, Y + Y_\Delta, \theta) = v(X, Y, \theta) + \left\langle \frac{[V(X, Y, \theta)]^+}{v(X, Y, \theta)}, V_\Delta \right\rangle + o(\|V_\Delta\|).$$

Next, we notice that

$$[V(X, Y, \theta)]^+ = \begin{bmatrix} [P(X, \theta)]^+ & 0 & 0 \\ 0 & [Q(Y, \theta)]^+ & 0 \\ 0 & 0 & [R(X, Y)]^+ \end{bmatrix}.$$

Finally, with standard trace computations, we derive

$$\begin{aligned} v(X + X_\Delta, Y + Y_\Delta, \theta) &= v(X, Y, \theta) + \mathrm{tr}\left\{ \frac{[P(X, \theta)]^+}{v(X, Y, \theta)} (A(\theta) + XC_1^{\mathrm{T}}(\theta)C_1(\theta)) \right. \\ &\quad + (A^{\mathrm{T}}(\theta) + C_1^{\mathrm{T}}(\theta)C_1(\theta)X) \frac{[P(X, \theta)]^+}{v(X, Y, \theta)} \\ &\quad \left. - [\, I \quad 0 \,] \frac{[R(X, Y)]^+}{v(X, Y, \theta)} \begin{bmatrix} I \\ 0 \end{bmatrix} \right\} X_\Delta \\ &\quad + \mathrm{tr}\left\{ \frac{[Q(Y, \theta)]^+}{v(X, Y, \theta)} (A^{\mathrm{T}}(\theta) + YB_1(\theta)B_1^{\mathrm{T}}(\theta)) \right. \\ &\quad + (A(\theta) + B_1(\theta)B_1^{\mathrm{T}}(\theta)Y) \frac{[Q(Y, \theta)]^+}{v(X, Y, \theta)} \\ &\quad \left. - [\, 0 \quad I \,] \frac{[R(X, Y)]^+}{v(X, Y, \theta)} \begin{bmatrix} 0 \\ I \end{bmatrix} \right\} Y_\Delta + o(\|V_\Delta\|). \end{aligned}$$

This concludes the proof of the lemma. □

We notice in Lemma 2 there are two decoupled QMIs $P(X, \theta) \leqslant 0$ and $Q(Y, \theta) \leqslant 0$, and the LMI $R(X, Y) \leqslant 0$ with coupling on X and Y. Hence, the subgradients computed in the lemma above are coupled through the term $[R(X, Y)]^+$. If $R(X, Y) \leqslant 0$, then $[R(X, Y)]^+ = 0$, thus the coupling terms in the subgradients

$$[\, I \quad 0 \,] \frac{[R(X, Y)]^+}{v(X, Y, \theta)} \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad [\, 0 \quad I \,] \frac{[R(X, Y)]^+}{v(X, Y, \theta)} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

disappear. In this case, each subgradient takes the same form of that computed in Polyak and Tempo (2001) for a linear quadratic regulator with state feedback.

3.2. *Randomized algorithm and probabilistic convergence*

To guarantee convergence in probability of the proposed algorithm, we need to state two mild assumptions. First, we assume that the problem defined by the solvability conditions (7)–(9) with fixed ε is strictly feasible.

Assumption 1. For fixed $\varepsilon > 0$, the feasible solution set, defined as

$$S_\Theta^\varepsilon \doteq \{(X = X^{\mathrm{T}},\ Y = Y^{\mathrm{T}}) \colon v(X, Y, \theta) = 0 \quad \forall \theta \in \Theta\}$$

has a non-empty interior.

We remark that, for $\varepsilon > 0$, $S_\Theta^\varepsilon \subset S_\Theta^0$, where S_Θ^0 is the original feasibility set of Lemma 2, defined as the set of $(X,\ Y)$ such that there exists $\varepsilon > 0$ satisfying (7)–(9). In the remaining of this paper, we suppose that such an appropriate ε is selected[2] and fixed.

Next, we introduce a probabilistic assumption regarding $v(X, Y, \theta)$.

Assumption 2. For any $(X,\ Y) \notin S_\Theta^\varepsilon$,

$$\mathrm{Prob}\{v(X, Y, \theta) > 0\} > 0. \tag{13}$$

This second assumption is related to the properties of the chosen pdf $f_\theta(\theta)$. In particular, we remark that Assumption 2 is automatically satisfied if Θ is a compact set and the probability density function $f_\theta(\theta)$ is nonzero for all $\theta \in \Theta$. To see this, recall that all matrices of the plant are continuous functions of θ. This means that $v(X, Y, \theta)$ is also a continuous function of θ for any $(X,\ Y)$. Thus, when Θ is a compact set, we see that the volume of θ satisfying $v(X, Y, \theta) > 0$ does not vanish for all $(X,\ Y) \notin S_\Theta^\varepsilon$. This fact guarantees that relation (13) holds whenever $f_\theta(\theta)$ is a probability density function such that $f_\theta(\theta) > 0$ for all $\theta \in \Theta$.

We now formally state the randomized algorithm.

[2] The existence of such an appropriate ε is guaranteed whenever S_Θ^0 has a nonempty interior.

Algorithm 1. *Define*

$$w(X,Y,\theta) \doteq (\|\partial_X\{v(X,Y,\theta)\}\|^2 + \|\partial_Y\{v(X,Y,\theta\}\|^2)^{1/2}.$$

Set initial matrices $X^0=(X^0)^{\mathrm{T}}$ *and* $Y^0=(Y^0)^{\mathrm{T}}$.

At each step k, *generate* θ^k *in the set* Θ *according to the pdf* $f_\theta(\theta)$ *and compute*

$$X^{k+1} = X^k - \mu^k \frac{\partial_X\{v(X^k,Y^k,\theta^k)\}}{w(X^k,Y^k,\theta^k)}, \tag{14}$$

$$Y^{k+1} = Y^k - \mu^k \frac{\partial_Y\{v(X^k,Y^k,\theta^k)\}}{w(X^k,Y^k,\theta^k)}, \tag{15}$$

if $v(X^k,Y^k,\theta^k)>0$ *or*

$$X^{k+1}=X^k, \qquad Y^{k+1}=Y^k$$

if $v(X^k,Y^k,\theta^k)=0$. *The stepsize* μ^k *is given by*

$$\mu^k \doteq \frac{v(X^k,Y^k,\theta^k)}{w(X^k,Y^k,\theta^k)} + r, \tag{16}$$

where the scalar parameter $r>0$ *is the radius of a ball* $B_r \subset S_\Theta^\varepsilon$.

Remark 1. The choice of the parameter step in stochastic gradient algorithms has been widely addressed in the literature, see, e.g., Kushner and Yin (1997). For example, a classical choice of the parameter μ^k in (14) and (15) may be

$$\mu^k \to 0, \qquad \sum_{k=0}^{\infty} \mu^k = \infty.$$

In this paper, we propose an alternative choice, which is based on additional information on the parameter $r>0$, whose existence is guaranteed by Assumption 2. If this information is not available, μ^k may be selected as discussed above.

The main result of this paper is the theorem which is presented next.

Theorem 4. *Let Assumptions* 1 *and* 2 *be satisfied. Then, Algorithm* 1 *converges with probability one in a finite number of iterations. That is,*

$$\mathrm{Prob}\{\exists k_0<\infty\colon (X^k,Y^k)\in S_\Theta^\varepsilon \subset S_\Theta^0 \ \forall k \geqslant k_0\} = 1.$$

Proof. Let

$$\bar{X} = X^* + r\,\frac{\partial_X\{v(X^k,Y^k,\theta^k)\}}{w(X^k,Y^k,\theta^k)},$$

$$\bar{Y} = Y^* + r\,\frac{\partial_Y\{v(X^k,Y^k,\theta^k)\}}{w(X^k,Y^k,\theta^k)},$$

where (X^*, Y^*) denotes the center of the ball B_r. Then, due to Assumption 1, $(\bar{X},\bar{Y})$ is a feasible solution in the set S_Θ^ε and, in particular, $v(\bar{X},\bar{Y},\theta^k)=0$. Now, if $v(X^k,Y^k,\theta^k)>0$, we write the following chain of equalities:

$$\begin{aligned}
&\|X^{k+1}-X^*\|^2 + \|Y^{k+1}-Y^*\|^2 \\
&\quad = \left\| X^k - X^* - \mu^k \frac{\partial_X\{v(X^k,Y^k,\theta^k)\}}{w(X^k,Y^k,\theta^k)} \right\|^2 \\
&\qquad + \left\| Y^k - Y^* - \mu^k \frac{\partial_Y\{v(X^k,Y^k,\theta^k)\}}{w(X^k,Y^k,\theta^k)} \right\|^2 \\
&\quad = \|X^k - X^*\|^2 \\
&\qquad - 2\mu^k \left\langle \frac{\partial_X\{v(X^k,Y^k,\theta^k)\}}{w(X^k,Y^k,\theta^k)}, (X^k-\bar{X}) \right\rangle \\
&\qquad - 2\mu^k \left\langle \frac{\partial_X\{v(X^k,Y^k,\theta^k)\}}{w(X^k,Y^k,\theta^k)}, (\bar{X}-X^*) \right\rangle \\
&\qquad + \|Y^k - Y^*\|^2 \\
&\qquad - 2\mu^k \left\langle \frac{\partial_Y\{v(X^k,Y^k,\theta^k)\}}{w(X^k,Y^k,\theta^k)}, (Y^k-\bar{Y}) \right\rangle \\
&\qquad - 2\mu^k \left\langle \frac{\partial_Y\{v(X^k,Y^k,\theta^k)\}}{w(X^k,Y^k,\theta^k)}, (\bar{Y}-Y^*) \right\rangle + (\mu^k)^2.
\end{aligned} \tag{17}$$

We now consider the inner product terms in the equalities above. Due to convexity of $v(X,Y,\theta)$ and to the fact that $(\bar{X},\bar{Y}) \in S_\Theta^\varepsilon$, we obtain

$$\begin{aligned}
&\langle \partial_X\{v(X^k,Y^k,\theta^k)\}, (X^k-\bar{X})\rangle \\
&\quad + \langle \partial_Y\{v(X^k,Y^k,\theta^k)\}, (Y^k-\bar{Y})\rangle \geqslant v(X^k,Y^k,\theta^k),
\end{aligned}$$

while, due to definition of $\bar{X}$ and $\bar{Y}$, it follows that

$$\begin{aligned}
&\left\langle \frac{\partial_X\{v(X^k,Y^k,\theta^k)\}}{w(X^k,Y^k,\theta^k)}, (\bar{X}-X^*) \right\rangle \\
&\quad + \left\langle \frac{\partial_Y\{v(X^k,Y^k,\theta^k)\}}{w(X^k,Y^k,\theta^k)}, (\bar{Y}-Y^*) \right\rangle = r.
\end{aligned}$$

Thus, we write

$$\begin{aligned}
&\|X^{k+1}-X^*\|^2 + \|Y^{k+1}-Y^*\|^2 \\
&\quad \leqslant \|X^k-X^*\|^2 + \|Y^k-Y^*\|^2 \\
&\qquad + (\mu^k)^2 - 2\mu^k \left(\frac{v(X^k,Y^k,\theta^k)}{w(X^k,Y^k,\theta^k)} + r \right).
\end{aligned}$$

Now, substituting the value of μ^k (16), we easily get

$$\begin{aligned}
&\|X^{k+1}-X^*\|^2 + \|Y^{k+1}-Y^*\|^2 \\
&\quad \leqslant \|X^k-X^*\|^2 + \|Y^k-Y^*\|^2 \\
&\qquad - \left(\frac{v(X^k,Y^k,\theta^k)}{w(X^k,Y^k,\theta^k)} + r \right)^2.
\end{aligned}$$

Observing that $v(X^k, Y^k, \theta^k)/w(X^k, Y^k, \theta^k) \geqslant 0$, it follows that

$$\left(\frac{v(X^k, Y^k, \theta^k)}{w(X^k, Y^k, \theta^k)} + r\right)^2 \geqslant r^2.$$

Hence, if $v(X^k, Y^k, \theta^k) > 0$, we obtain

$$\|X^{k+1} - X^*\|^2 + \|Y^{k+1} - Y^*\|^2 \leqslant \|X^k - X^*\|^2 + \|Y^k - Y^*\|^2 - r^2. \tag{18}$$

From this formula, we conclude that no more than $\lceil(\|X^0 - X^*\|^2 + \|Y^0 - Y^*\|^2)/r^2\rceil$ correction steps should be executed. On the other hand, if $(X^k, Y^k) \notin S_\Theta^\varepsilon$, then, due to Assumption 2, there is a nonzero probability to make a correction step. Thus, with probability one, the method cannot terminate at an infeasible point. We therefore conclude that the algorithm converges to a solution in the set $S_\Theta^\varepsilon \subset S_\Theta^0$ in a finite number of steps. □

Remark 2. In Algorithm 1, we can use another iteration formula with a projection

$$X^{k+1} = \left[X^k - \mu^k \frac{\partial_X\{v(X^k, Y^k, \theta^k)\}}{w(X^k, Y^k, \theta^k)}\right]^+, \tag{19}$$

$$Y^{k+1} = \left[Y^k - \mu^k \frac{\partial_Y\{v(X^k, Y^k, \theta^k)\}}{w(X^k, Y^k, \theta^k)}\right]^+ \tag{20}$$

instead of (14) and (15). In fact, Theorem 4 still holds because equality (17) can be replaced by

$$\begin{aligned}
&\|X^{k+1} - X^*\|^2 + \|Y^{k+1} - Y^*\|^2 \\
&= \left\|\left[X^k - \mu^k \frac{\partial_X\{v(X^k, Y^k, \theta^k)\}}{w(X^k, Y^k, \theta^k)}\right]^+ - X^*\right\|^2 \\
&\quad + \left\|\left[Y^k - \mu^k \frac{\partial_Y\{v(X^k, Y^k, \theta^k)\}}{w(X^k, Y^k, \theta^k)}\right]^+ - Y^*\right\|^2 \\
&\leqslant \left\|X^k - X^* - \mu^k \frac{\partial_X\{v(X^k, Y^k, \theta^k)\}}{w(X^k, Y^k, \theta^k)}\right\|^2 \\
&\quad + \left\|Y^k - Y^* - \mu^k \frac{\partial_Y\{v(X^k, Y^k, \theta^k)\}}{w(X^k, Y^k, \theta^k)}\right\|^2.
\end{aligned} \tag{21}$$

The above inequality suggests us that the projections in (19) and (20) in Algorithm 1 may improve the convergence with additional computation. A related comment can be found in Remark 4.

This theorem is an extension for LPV systems of the main theorem in Polyak and Tempo (2001). The proof of this result follows along the same lines, with the difference that in Algorithm 1 two sequences are simultaneously considered. Theorem 4 guarantees probabilistic convergence of the algorithm. Clearly, since this is a randomized algorithm, we need to set in advance the number of iterations when we actually perform the algorithm. Thus, further information to decide this number is needed. In this regard, we note that a lower bound of the probability such that $(X^k, Y^k) \in S_\Theta^0$ for a fixed k can be computed. To address this issue, define m as

$$m \doteq \left\lceil \frac{\|X^0 - X^*\|^2 + \|Y^0 - Y^*\|^2}{r^2} \right\rceil, \tag{22}$$

where r, X^*, and Y^* are given in the proof of Theorem 4. Subsequently, suppose that there exists a lower bound p on the probability of update such that

$$\text{Prob}\{v(X, Y, \theta) > 0\} \geqslant p > 0 \tag{23}$$

for all $(X, Y) \notin S_\Theta^0$. Note that p always exists when Θ is a compact set and $f_\theta(\theta) > 0$ for all $\theta \in \Theta$. This can be shown recalling (13) in Assumption 2, together with the observation that the QMIs (7) and (8) guarantee that every boundary point of S_Θ^0 is an exterior point of S_Θ^ε. Next, applying the one-sided Chernoff bound for $k > m/p$, we conclude that

$$\text{Prob}\{(X^k, Y^k) \in S_\Theta^0\} \geqslant 1 - \exp(-2(pk - m)^2/k). \tag{24}$$

Even though m and p may be difficult to estimate, the bound above provides information on the rate of convergence. That is, the risk associated with (X^k, Y^k) decreases exponentially as k tends to infinity, and it is independent of the dimension of the vector θ. Moreover, using the same lower bound p on the probability of update, we see that the expectation of the number $k_{\max}$ of iterations required by the algorithm is bounded as

$$\mathscr{E}[k_{\max}] \leqslant \sum_{k=m}^{\infty} k \begin{pmatrix} k-1 \\ m-1 \end{pmatrix} p^m (1-p)^{k-m} = \frac{m}{p}. \tag{25}$$

Thus, we immediately conclude that the expected number of iterations $\mathscr{E}[k_{\max}]$ is finite. This comment follows from Oishi and Kimura (2001), where a similar randomized algorithm is studied and the expected number of iterations for decreasing values of ε is investigated.

Finally, we observe that even if m and p are not available, Algorithm 1 still produces a "good feasible candidate," provided that the number of iterations is sufficiently large. Subsequently, for finite families the user can always check if this probabilistic candidate is also a guaranteed solution—obviously, this check is much easier to perform than actually finding a solution.

3.3. *Further properties: boundedness and monotone-like convergence*

In this subsection, we study the properties of the norm of the solution (X^k, Y^k). In particular, Theorem 4 guarantees that the algorithm terminates in a finite number of iterations. This implies that the norm of the final solution never becomes infinity even if the feasibility set S_Θ^ε includes a member whose norm is arbitrarily large. In the next corollary, we show that the norm of the solution (X^k, Y^k) is bounded independently of the number of iterations.

Corollary 5. *Let Assumptions* 1 *and* 2 *be satisfied. Then, the norm of the solutions* (X^k, Y^k) *given by Algorithm* 1 *is bounded independently of k. That is, the inequality*

$$\begin{aligned}(\|X^k\|^2 + \|Y^k\|^2)^{1/2} &\leqslant (\|X^0\|^2 + \|Y^0\|^2)^{1/2} \\ &\quad + \inf_{(\tilde{X},\tilde{Y})\in S_\Theta^\varepsilon} 2(\|\tilde{X}\|^2 + \|\tilde{Y}\|^2)^{1/2} \end{aligned} \tag{26}$$

holds for all $k \geqslant 0$. *Moreover, the inequality*

$$\begin{aligned}(\|X^{k+1}\|^2 + \|Y^{k+1}\|^2)^{1/2} &\leqslant (\|X^k\|^2 + \|Y^k\|^2)^{1/2} \\ &\quad + \inf_{(\tilde{X},\tilde{Y})\in S_\Theta^\varepsilon} 2(\|\tilde{X}\|^2 + \|\tilde{Y}\|^2)^{1/2} \end{aligned} \tag{27}$$

holds for all $k \geqslant 0$.

Proof. From the proof of Theorem 4, we conclude that if $v(X^k, Y^k, \theta^k) > 0$ then inequality (18) holds for any $r > 0$. On the other hand, if $v(X^k, Y^k, \theta^k) = 0$ then $X^{k+1} = X^k$ and $Y^{k+1} = Y^k$. Thus, it follows that

$$\begin{aligned}(\|X^{k+1} - \tilde{X}\|^2 + \|Y^{k+1} - \tilde{Y}\|^2)^{1/2} \\ \leqslant (\|X^k - \tilde{X}\|^2 + \|Y^k - \tilde{Y}\|^2)^{1/2}\end{aligned}$$

holds for all $k \geqslant 0$ and for all $(\tilde{X}, \tilde{Y}) \in S_\Theta^\varepsilon$. This immediately implies that

$$\begin{aligned}(\|X^k - \tilde{X}\|^2 + \|Y^k - \tilde{Y}\|^2)^{1/2} \\ \leqslant (\|X^0 - \tilde{X}\|^2 + \|Y^0 - \tilde{Y}\|^2)^{1/2}\end{aligned}$$

holds for all $k \geqslant 0$ and for all $(\tilde{X}, \tilde{Y}) \in S_\Theta^\varepsilon$. Using this fact and standard Frobenius norm properties, we compute the bounds

$$\begin{aligned}(\|X^k\|^2 + \|Y^k\|^2)^{1/2} &= (\|X^k - \tilde{X} + \tilde{X}\|^2 + \|Y^k - \tilde{Y} + \tilde{Y}\|^2)^{1/2} \\ &\leqslant (\|X^k - \tilde{X}\|^2 + \|Y^k - \tilde{Y}\|^2)^{1/2} + (\|\tilde{X}\|^2 + \|\tilde{Y}\|^2)^{1/2} \\ &\leqslant (\|X^0 - \tilde{X}\|^2 + \|Y^0 - \tilde{Y}\|^2)^{1/2} + (\|\tilde{X}\|^2 + \|\tilde{Y}\|^2)^{1/2} \\ &\leqslant (\|X^0\|^2 + \|Y^0\|^2)^{1/2} + 2(\|\tilde{X}\|^2 + \|\tilde{Y}\|^2)^{1/2}\end{aligned}$$

for all $(\tilde{X}, \tilde{Y}) \in S_\Theta^\varepsilon$. Therefore, we conclude that inequality (26) holds. The derivation of (27) proceeds along similar lines. □

The corollary says that the upper bound of the norm of (X^k, Y^k) is determined by a feasible solution having the smallest norm and is independent of k. In addition, the corollary says that, at each iteration, the norm of (X^k, Y^k) does not increase beyond a constant which is determined independently of k. This fact implies a monotone-like convergence.

3.4. Computational complexity: finite families

We remark that further insight on computational complexity can be obtained if the scheduling vector θ assumes only a finite number of values, i.e., if Θ consists of a finite family of the form

$$\Theta = \{\theta^1, \theta^2, \ldots, \theta^\ell\}.$$

This situation may arise, for example, in the classical setting of Becker and Packard (1994) if Θ is a hypercube and its vertices are considered. If we assign the same probability $1/\ell$ at each vertex θ^i, $i = 1, 2, \ldots, \ell$, we see that the probability p defined in (23) is given by $p = 1/\ell$. Thus, we conclude that bound (25) of the average number of iterations is

$$\mathscr{E}[k_{\max}] \leqslant \ell m. \tag{28}$$

The meaning of this upper bound becomes clear when we consider a "derandomization" approach as discussed in Mulmuley (1994). To this end, we select elements in Θ cyclically according to a fixed order. Then, we obtain a deterministic version of Algorithm 1, which is related to the algorithm stated in Bondarko and Yakubovich (1992). For this deterministic version, it can be easily shown that the number of iterations is bounded by the same value ℓm, because at least a correction step occurs in each cycle. That is, in the worst-case, the average number of iterations coincides with its deterministic counterpart.

Finally, we note that the number ℓ may depend on the dimension of θ exponentially, as in the case considered in Becker and Packard (1994). Thus, in the sequential algorithm studied in this paper, we see that the issue of computational complexity appears in the worst-case number of iterations, and may be a critical issue. On the other hand, in the classical LMI formulation of LPV problems, see, e.g., Becker and Packard (1994), and Apkarian and Adams (1998), computational complexity issues arise in the number of LMIs that have to be solved *simultaneously* and this requires heavy computational effort.

4. Numerical examples

4.1. Lateral motion of an aircraft

To show the effectiveness of our technique to deal with a large number of scheduling parameters, we present a numerical example. This example is a modified version of the multivariable example given in Polyak and Tempo (2001), see also the original paper Tyler and Tuteur (1966) which studies the state feedback design for the lateral motion of an aircraft.

The model consists of four states and two inputs. The state space equations are given by

$$\dot{x}(t) = A(\theta(t))x(t) + B_2u(t), \quad y(t) = C_2x(t),$$

where

$$A(\theta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \theta_1 & \theta_2 & \theta_3 \\ \theta_4 & 0 & \theta_5 & -1 \\ \theta_4\theta_6 & \theta_7 & \theta_8 + \theta_5\theta_6 & \theta_9 - \theta_6 \end{bmatrix},$$

$$B_2(\theta) = \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In Tyler and Tuteur (1966), the inputs u_1 and u_2 represent the rudder and aileron deflections, while the state variables x_1, x_2, x_3, x_4 are the bank angle, its derivative, the side-slip angle, and the yaw rate, respectively. Here, we selected as outputs the variables x_1, x_3, x_4.

The vector of scheduling parameter θ is allowed to vary 10% around its nominal value $\bar{\theta} = [-2.93, -4.75, 0.78, 0.086, -0.11, 0.1, -0.042, 2.601, -0.29]^T$, i.e.,

$$\theta \in \Theta = \{\theta \in \mathbb{R}^9\colon \theta_i \in [0.9\bar{\theta}_i, 1.1\bar{\theta}_i],\ i = 1, 2, \ldots, 9\}.$$

We assume that the pdf of θ is uniform on Θ. Finally, we set $\gamma = 3$ and select

$$B_1 = [\,0.1\,B_2 \quad 0\,], \quad D_{21} = [\,0 \quad I\,],$$

$$C_1 = \begin{bmatrix} 0.1\,C_2 \\ 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

The scheduling parameters θ enter into the state space equation in a multi-affine manner. Therefore, classical quadratic L_2 performance can be detected solving $2^9 = 512$ LMIs simultaneously. We remark that this could be a computationally difficult task.

We now describe the results of the simulations. First, we randomly generated initial conditions of the form

$$X^0 = \begin{bmatrix} 0.458026 & -0.154166 & 0.164914 & 0.054525 \\ -0.154166 & 0.122168 & -0.141305 & 0.076440 \\ 0.164914 & -0.141305 & 0.250909 & 0.038064 \\ 0.054525 & 0.076440 & 0.038064 & 0.471667 \end{bmatrix},$$

$$Y^0 = \begin{bmatrix} 0.311724 & -0.088885 & -0.102318 & 0.342056 \\ -0.088885 & 0.350925 & -0.165670 & -0.133283 \\ -0.102318 & -0.165670 & 0.337603 & -0.213017 \\ 0.342056 & -0.133283 & -0.213017 & 0.473112 \end{bmatrix}.$$

Then, setting $r = 0.001$, we sequentially generated 1000 random vectors θ^k obtaining a sequence of solutions X^k and Y^k according to Algorithm 1. Choosing $\varepsilon = 0.08$, 28 updates occurred and the final solutions X^{1000} and Y^{1000} were given by

$$X^{1000} = \begin{bmatrix} 0.525869 & -0.088285 & 0.125412 & -0.019091 \\ -0.088285 & 0.354168 & -0.068721 & 0.015752 \\ 0.125412 & -0.068721 & 0.333923 & 0.114620 \\ -0.019091 & 0.015752 & 0.114620 & 0.514689 \end{bmatrix},$$

$$Y^{1000} = \begin{bmatrix} 0.415584 & 0.049482 & -0.126227 & 0.164431 \\ 0.049482 & 0.347744 & 0.033614 & 0.013021 \\ -0.126227 & 0.033614 & 0.518057 & -0.222386 \\ 0.164431 & 0.013021 & -0.222386 & 0.449475 \end{bmatrix}.$$

These solutions satisfy all 512 equations given by the vertex set and therefore quadratic L_2 performance in worst-case sense is achieved.

Next, we increased ε to the value 0.2, and we used the same random sequence. Then, 50 updates were observed and similar final solutions were obtained. These solutions also satisfied all 512 equations. On the other hand, when we decreased ε to 0.001, 18 updates occurred and the obtained solutions did not satisfy 96 equations. We observed that the selection of larger values of ε corresponds to an increase of the lower bound on the probability of update p defined in (23), since the volume of θ satisfying $v(X, Y, \theta) > 0$ becomes greater for all $(X, Y) \notin S^0_\Theta$. On the other hand, it should be noted that, as ε increases, the feasible set S^ε_Θ shrinks. This limits the selection of r in Eq. (16) and therefore the size of the update step. The tradeoff between the number of updates and the size of each update suggests a way to select ε and r in practice.

4.2. A two-link flexible manipulator

In this subsection, we study a model of a two-link flexible manipulator, described in Apkarian and Adams (1998), with some suitable modifications.

The model of the manipulator is given by

$$M(\alpha_2(t))\ddot{q}(t) + D\dot{q}(t) + Kq(t) = Fu_c(t),$$

$$y_m(t) = F^T q(t),$$

where

$$q(t) = \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \end{bmatrix}$$

and

$$M(\alpha_2) = M(\pi/2) + \cos\alpha_2\{M(\pi/2) - M(\pi)\},$$

$$M(\pi/2) = \begin{bmatrix} 34.7077 & 9.7246 & 23.6398 & 5.9114 \\ 9.7246 & 9.8783 & 9.7246 & 5.9114 \\ 23.6398 & 9.7246 & 17.5711 & 5.9114 \\ 5.9114 & 5.9114 & 5.9114 & 3.7233 \end{bmatrix},$$

$$M(\pi) = \begin{bmatrix} 17.0296 & 0.8856 & 9.7776 & 0.8430 \\ 0.8856 & 9.8783 & 4.7016 & 5.9114 \\ 9.7776 & 4.7016 & 7.5249 & 3.0311 \\ 0.8430 & 5.9114 & 3.0311 & 3.7233 \end{bmatrix},$$

$$D = \mathrm{diag}\{d_1, d_2, 0.09, 0.05\},$$

$$K = \mathrm{diag}\{k_1, k_2, 89.1473, 45.6434\},$$

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T.$$

In these equations, $\alpha_1(t)$ and $\alpha_2(t)$ are the shoulder and the elbow joint angles, $u_c(t)$ is the torque control input, and $y_m(t)$ is the angle measurement. The elbow joint angle $\alpha_2(t) \in [0, \pi]$ is treated as the scheduling parameter

$$\theta(t) \doteq \alpha_2(t).$$

In Apkarian and Adams (1998) the values $d_1=d_2=k_1=k_2=0$ are studied, while here we consider $d_1 = d_2 = 3$, $k_1 = k_2 = 5$. This corresponds to the introduction of a low authority control of the form

$$u_c(t) = -3\dot{y}_m(t) - 5y_m(t) + u(t)$$

that stabilizes the rigid modes of the plant. This control has the property to improve the numerical conditioning of the problem.

The generalized LPV plant is constructed in the same way as in Apkarian and Adams (1998). First, the state space representation of the plant is given as

$$\begin{bmatrix} A_g(\theta) & B_g(\theta) \\ C_g & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & I & 0 \\ -M(\theta)^{-1}K & -M(\theta)^{-1}D & M(\theta)^{-1}F \\ \hline F^T & 0 & 0 \end{array}\right].$$

Next, to cope with the flexible modes of the plant, we introduce weighting functions

$$\begin{bmatrix} A_{wf} & B_{wf} \\ C_{wf} & D_{wf} \end{bmatrix} = \left[\begin{array}{cccc|cc} -100 & -99.9 & 0 & 0 & -99.9 & 0 \\ 0 & -100 & 0 & 0 & -99.9 & 0 \\ 0 & 0 & -100 & -99.9 & 0 & -99.9 \\ 0 & 0 & 0 & -100 & 0 & -99.9 \\ \hline 4 & 4 & 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 4 & 0 & 4 \end{array}\right]$$

$$\begin{bmatrix} A_{wp} & B_{wp} \\ C_{wp} & D_{wp} \end{bmatrix} = \left[\begin{array}{cc|cc} -0.03 & 0 & 1.036 & 0 \\ 0 & -0.03 & 0 & 1.036 \\ \hline 0.1075 & 0 & 0.1075 & 0 \\ 0 & 0.1075 & 0 & 0.1075 \end{array}\right].$$

Then, the generalized LPV plant is obtained as

$$\begin{bmatrix} A(\theta) & B_1 & B_2(\theta) \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} = \left[\begin{array}{ccc|cc|c} A_{wp} & 0 & B_{wp}C_g & B_{wp} & 0 & 0 \\ 0 & A_{wf} & 0 & 0 & 0 & B_{wf} \\ 0 & 0 & A_g(\theta) & 0 & 0 & B_g(\theta) \\ \hline C_{wp} & 0 & D_{wp}C_g & D_{wp} & 0 & 0 \\ 0 & C_{wf} & 0 & 0 & 0 & D_{wf} \\ \hline 0 & 0 & C_g & I & I & 0 \end{array}\right].$$

Finally, a balanced realization of the generalized LPV plant is used, setting θ to $\pi/2$ and computing the corresponding state transformation. We note that the balanced realization can be used for stable plants, and the low authority control guarantees stability of the generalized LPV plant.

Then, we executed the randomized algorithm described in Appendix A to solve the general quadratic L_2

control problem. According to Case 1 in Apkarian and Adams (1998), we set n_p to one and assumed the scheduled variable θ *frozen* in time [3] setting $\dot{\theta}(t)=0$. In this case, the general formulas given in Appendix A are specialized setting $X_d(\theta,\tilde{\theta})=0$, $Y_d(\theta,\tilde{\theta})=0$ in (A.3) and (A.4). The function $\rho(\theta)$ defined in (A.2) was chosen as $\rho(\theta)\doteq\cos\theta$ and the L_2 norm specification was selected as $\gamma=4$. The initial conditions were generated randomly, and 30 000 random samples of θ were generated sequentially according to a uniform distribution in the interval [0, π]. The parameters ε and r were set to 0.008 and 0.001, respectively. We observed that 8, 143 updates occurred and we obtained a final solution satisfying the QMIs at 1000 equispaced points for θ.

5. Concluding remarks

In this paper, we studied probabilistic design of LPV control systems. The main result given here is to prove probabilistic convergence of a gradient-based randomized algorithm. That is, the proposed algorithm provides a "probabilistic feasible solution" and has a finite termination property. The algorithm also enjoys additional interesting properties, such as monotone-like convergence.

Throughout this paper, we have emphasized that randomized algorithms are viable tools to treat nonlinearities on the scheduling parameter. We note that, in the second numerical example, the scheduling parameter is one of the state variables, and we generated its values randomly. The idea of "randomly generating the state variables" leads to a research direction for general nonlinear systems which is now briefly described.

We consider a nonlinear system

$$\dot{x}(t)=f(x(t)), \tag{29}$$

where the state $x(t)\in\mathbb{R}^n$ is constrained within the set $\mathbf{X}\subset\mathbb{R}^n$. We assume that this nonlinear system has a unique solution for all $x(0)\in\mathbf{X}$, and $f(x)$ is uniquely determined for all $x\in\mathbf{X}$.

For simplicity, we consider quadratic stability analysis with a candidate Lyapunov function of the form $L(x)\doteq x^{\mathrm{T}}Px$, where $P=P^{\mathrm{T}}>0$. The time derivative of the function $L(x)$ is given by

$$\frac{\mathrm{d}L(x(t))}{\mathrm{d}t}=2x^{\mathrm{T}}(t)Pf(x(t)).$$

Then, the problem is to find $P=P^{\mathrm{T}}>0$ satisfying

$$2x^{\mathrm{T}}Pf(x)+\varepsilon x^{\mathrm{T}}x\leqslant 0$$

for all $x\in\mathbf{X}$, where ε is a small positive scalar. If we find such P, then we conclude that (29) is stable for all $x(0)\in\{x\colon x^{\mathrm{T}}Px\leqslant\rho,\rho>0\}\subset\mathbf{X}$.

We notice that this problem can be handled with the techniques developed in this paper. That is, x and P can be regarded as θ and (X,Y), respectively. Since $2x^{\mathrm{T}}Pf(x)+\varepsilon$ is convex in P, we see that a randomized algorithm similar to Algorithm 1, based on random generation of x within the set $\mathbf{X}$, can be easily developed to determine probabilistic stability of nonlinear constrained systems. We feel that probabilistic design of stabilizing controllers for nonlinear constrained systems is a promising area of future research.

[3] This corresponds to consider slow time-varying systems in which the time variation of the parameter is much slower than the system dynamics.

Appendix A. General case

In this appendix, we present the formulation of the general LPV problem, obtained removing the regularity assumptions introduced in Section 2. That is, we consider the case when assumptions (a) and (b) are not satisfied and we replace (c) with the assumption that the vector $\theta(t)$ is a continuous function of t and $\dot{\theta}(t)\in\tilde{\Theta}\subset\mathbb{R}^{n_s}$ for all $t>0$. Then, we suppose that on-line measurements of the scheduling parameter and its derivative are available.

Following the same developments, we first formally state the LPV L_2 control problem in this general setting. We consider here the plant and controller description defined in Eqs. (1) and (3) with $A_c(\theta(t),\dot{\theta}(t))$ and $A_{cl}(\theta(t),\dot{\theta}(t))$ instead of $A_c(\theta(t))$ and $A_{cl}(\theta(t))$. For the closed-loop system (4), we introduce $D_{cl}(\theta)=D_{11}(\theta)$.

Problem A.1 (General LPV L_2 control). Given $\gamma>0$, find $A_c(\theta,\tilde{\theta})$, $B_c(\theta)$, and $C_c(\theta)$ such that there exist a continuously differentiable matrix function $X_{cl}(\theta)=X_{cl}^{\mathrm{T}}(\theta)\in\mathbb{R}^{2n\times 2n}$ and a constant $\varepsilon>0$ satisfying

$$-\sum_{j=1}^{n_s}\frac{\partial X_{cl}(\theta)}{\partial\theta_j}\tilde{\theta}_j+A_{cl}(\theta)X_{cl}(\theta)+X_{cl}(\theta)A_{cl}^{\mathrm{T}}(\theta)$$
$$+\left[X_{cl}(\theta)C_{cl}^{\mathrm{T}}(\theta)\quad B_{cl}(\theta)\right]\Upsilon_{cl}^{-1}(\theta)\begin{bmatrix}C_{cl}(\theta)X_{cl}(\theta)\\ B_{cl}^{\mathrm{T}}(\theta)\end{bmatrix}$$
$$+\varepsilon I\leqslant 0,$$
$$X_{cl}(\theta)\geqslant 0, \tag{A.1}$$

for all $\theta\in\Theta$ and $\tilde{\theta}\in\tilde{\Theta}$, where

$$\Upsilon_{cl}(\theta)\doteq\begin{bmatrix}I & -D_{cl}(\theta)\\ -D_{cl}^{\mathrm{T}}(\theta) & \gamma^2 I\end{bmatrix}.$$

To tackle this problem, we consider the following matrix variables:

$$X(\theta)\doteq\sum_{i=0}^{n_p}\rho_i(\theta)X_i,\qquad Y(\theta)\doteq\sum_{i=0}^{n_p}\rho_i(\theta)Y_i,$$
$$G(\theta)\doteq\sum_{i=0}^{n_p}\rho_i(\theta)G_i,\qquad L(\theta)\doteq\sum_{i=0}^{n_p}\rho_i(\theta)L_i, \tag{A.2}$$

where $\rho_0(\theta) \equiv 1$ and, for each $i = 1, \ldots, n_p$, $\rho_i(\theta)$ is a differentiable function of θ, X_i, $Y_i \in \mathbb{R}^{n\times n}$, $G_i \in \mathbb{R}^{n_u\times n}$, and $L_i \in \mathbb{R}^{n\times n_y}$.

The solvability conditions for the general LPV L_2 problem are given in the following lemma. This lemma is a slight variation of a result given in Apkarian and Adams (1998).

Lemma A.1. *The LPV L_2 control problem is solvable if there exist continuously differentiable matrix functions $X(\theta)=X^{\mathrm{T}}(\theta)\in\mathbb{R}^{n\times n}$, $Y(\theta)=Y^{\mathrm{T}}(\theta)\in\mathbb{R}^{n\times n}$, and a constant $\varepsilon>0$ satisfying*

$$\begin{aligned}
&P_g(X, G, \theta, \tilde{\theta})\\
&\doteq -X_d(\theta,\tilde{\theta}) + A(\theta)X(\theta) + X(\theta)A^{\mathrm{T}}(\theta)\\
&\quad + B_2(\theta)G(\theta) + G^{\mathrm{T}}(\theta)B_2^{\mathrm{T}}(\theta)\\
&\quad + [\, X(\theta)C_1^{\mathrm{T}}(\theta) + G^{\mathrm{T}}(\theta)D_{12}^{\mathrm{T}}(\theta) \quad B_1(\theta)\,]\\
&\quad \times \Upsilon_X^{-1}(\theta)\begin{bmatrix} C_1(\theta)X(\theta) + D_{12}(\theta)G(\theta)\\ B_1^{\mathrm{T}}(\theta)\end{bmatrix} + \varepsilon I\\
&\leqslant 0,
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
&Q_g(Y, L, \theta, \tilde{\theta})\\
&\doteq Y_d(\theta,\tilde{\theta}) + A^{\mathrm{T}}(\theta)Y(\theta) + Y(\theta)A(\theta)\\
&\quad + C_2^{\mathrm{T}}(\theta)L^{\mathrm{T}}(\theta) + L(\theta)C_2(\theta)\\
&\quad + [\, Y(\theta)B_1(\theta) + L(\theta)D_{21}(\theta) \quad C_1^{\mathrm{T}}(\theta)\,]\\
&\quad \times \Upsilon_Y^{-1}(\theta)\begin{bmatrix} B_1^{\mathrm{T}}(\theta)Y(\theta) + D_{21}^{\mathrm{T}}(\theta)L^{\mathrm{T}}(\theta)\\ C_1(\theta)\end{bmatrix} + \varepsilon I\\
&\leqslant 0,
\end{aligned} \tag{A.4}$$

$$R_g(X, Y, \theta) \doteq -\begin{bmatrix} X(\theta) & \gamma^{-1}I\\ \gamma^{-1}I & Y(\theta)\end{bmatrix} + \varepsilon I \leqslant 0 \tag{A.5}$$

for all $\theta\in\Theta$ and $\tilde{\theta}\in\tilde{\Theta}$, where

$$\Upsilon_X(\theta) \doteq \begin{bmatrix} I & -D_{11}(\theta)\\ -D_{11}^{\mathrm{T}}(\theta) & \gamma^2 I\end{bmatrix},$$

$$\Upsilon_Y(\theta) \doteq \begin{bmatrix} I & -D_{11}^{\mathrm{T}}(\theta)\\ -D_{11}(\theta) & \gamma^2 I\end{bmatrix}$$

and

$$\begin{aligned}
X_d(\theta,\tilde{\theta}) &\doteq \sum_{j=0}^{n_s} \frac{\partial X(\theta)}{\partial \theta_j}\tilde{\theta}_j\\
&= \sum_{i=0}^{n_p}\left(\sum_{j=0}^{n_s}\frac{\partial \rho_i(\theta)}{\partial\theta_j}\tilde{\theta}_j\right)X_i \doteq \sum_{i=0}^{n_p}\tilde{\rho}_i(\theta,\tilde{\theta})X_i,\\
Y_d(\theta,\tilde{\theta}) &\doteq \sum_{j=0}^{n_s} \frac{\partial Y(\theta)}{\partial \theta_j}\tilde{\theta}_j\\
&= \sum_{i=0}^{n_p}\left(\sum_{j=0}^{n_s}\frac{\partial \rho_i(\theta)}{\partial\theta_j}\tilde{\theta}_j\right)Y_i. \doteq \sum_{i=0}^{n_p}\tilde{\rho}_i(\theta,\tilde{\theta})Y_i.
\end{aligned}$$

Furthermore, if the above conditions hold, the matrices of an LPV controller are given by

$$\begin{aligned}
&A_c(\theta,\dot{\theta})\\
&\quad = A(\theta) + Y^{-1}(\theta)L(\theta)C_2(\theta)\\
&\qquad + B_2(\theta)G(\theta)(X - \gamma^{-2}Y^{-1})^{-1}\\
&\qquad + [\,\gamma^{-2}Y^{-1}(\theta)C_1^{\mathrm{T}}(\theta) \quad B_1(\theta) + Y^{-1}(\theta)L(\theta)D_{21}(\theta)\,]\\
&\qquad \times \Upsilon_X^{-1}(\theta)\\
&\qquad \times \begin{bmatrix} C_1(\theta) + D_{12}(\theta)G(\theta)(X(\theta) - \gamma^{-2}Y^{-1}(\theta))^{-1}\\ -D_{21}^{\mathrm{T}}(\theta)L^{\mathrm{T}}(\theta)(X(\theta)Y(\theta) - \gamma^{-2}I)^{-1}\end{bmatrix}\\
&\qquad - \gamma^{-2}Y^{-1}(\theta)C_2^{\mathrm{T}}(\theta)L^{\mathrm{T}}(\theta)(X(\theta)Y(\theta) - \gamma^{-2}I)^{-1}\\
&\qquad + \gamma^{-2}Y^{-1}(\theta)(Q_g(Y, L, \theta, \dot{\theta}) - \varepsilon I)\\
&\qquad \times (X(\theta)Y(\theta) - \gamma^{-2}I)^{-1},\\
&B_c(\theta) = -Y^{-1}(\theta)L(\theta),\\
&C_c(\theta) = G(\theta)(X(\theta) - \gamma^{-2}Y^{-1}(\theta))^{-1}.
\end{aligned} \tag{A.6}$$

Remark A.1. We remark that, in the general case, Lemma 7 gives only sufficient conditions for the solvability of Problem 6. This is due to the fact that the search for solutions of (A.1) is restricted to those defined by (A.2).

To recast the feasibility conditions as a minimization problem, we introduce the matrix-valued function

$$\begin{aligned}
&V_g(X, Y, G, L, \theta, \tilde{\theta})\\
&\doteq \begin{bmatrix} P_g(X, G, \theta, \tilde{\theta}) & 0 & 0\\ 0 & Q_g(Y, L, \theta, \tilde{\theta}) & 0\\ 0 & 0 & R_g(X, Y, \theta)\end{bmatrix}
\end{aligned}$$

and define the scalar function

$$v_g(X,Y,G,L,\theta,\tilde\theta) \doteq \|[V_g(X,Y,G,L,\theta,\tilde\theta)]^+\|.$$

The function $v_g(\cdot)$ plays the same role of objective function played by $v(\cdot)$ for the regularized problem. In the following lemma some properties of $v_g(\cdot)$ are stated without proof.

Lemma A.2. *The function $v_g(X,Y,G,L,\theta,\tilde\theta)$ is convex in (X,Y,G,L) and, for $i=0,\dots,n_p$, its subgradients*

$$\partial_{X_i}\{v_g\} \doteq \partial_{X_i}\{v_g(X,Y,G,L,\theta,\tilde\theta)\},$$

$$\partial_{Y_i}\{v_g\} \doteq \partial_{Y_i}\{v_g(X,Y,G,L,\theta,\tilde\theta)\},$$

$$\partial_{G_i}\{v_g\} \doteq \partial_{G_i}\{v_g(X,Y,G,L,\theta,\tilde\theta)\},$$

$$\partial_{L_i}\{v_g\} \doteq \partial_{L_i}\{v_g(X,Y,G,L,\theta,\tilde\theta)\}$$

are given by Eqs. (A.7), (A.8), (A.9), (A.10), *if* $v_g(X,Y,G,L,\theta,\tilde\theta)>0$, *or*

$$\partial_{X_i}\{v_g\}=0,\quad \partial_{Y_i}\{v_g\}=0,$$

$$\partial_{G_i}\{v_g\}=0,\quad \partial_{L_i}\{v_g\}=0$$

if $v_g(X,Y,G,L,\theta,\tilde\theta)=0$.

In order to develop a randomized algorithm, the following assumptions are needed.

Assumption A.1. For fixed $\varepsilon>0$, the feasible solution set, defined as

$$S^\varepsilon_{\Theta,\tilde\Theta} \doteq \{(X=X^{\mathrm T},\ Y=Y^{\mathrm T},\ G,\ L):$$
$$v_g(X,Y,G,L,\theta,\tilde\theta)=0 \quad \forall\theta\in\Theta, \forall\tilde\theta\in\tilde\Theta\}$$

has a nonempty interior.

Assumption A.2. For any $(X,\ Y,\ G,\ L)\notin S^\varepsilon_{\Theta,\tilde\Theta}$, $\mathrm{Prob}\{v_g(X,Y,G,L,\theta,\tilde\theta)>0\}>0$.

Here we also assume that $\tilde\theta$ is a random vector with given pdf $f_{\tilde\theta}(\tilde\theta)$ on the support $\tilde\Theta$.

The algorithm for the general case is a straightforward modification of that given in Section 3 and is reported here for completeness.

Algorithm A.1. *Define*

$$w_g(X,Y,G,L,\theta,\tilde\theta) \doteq \left[\sum_{i=1}^{n_p}(\|\partial_{X_i}\{v_g\}\|^2+\|\partial_{Y_i}\{v_g\}\|^2 + \|\partial_{G_i}\{v_g\}\|^2+\|\partial_{L_i}\{v_g\}\|^2)\right]^{1/2}.$$

Subgradients equations of Lemma A.2

$$\partial_{X_i}\{v_g\} = \rho_i(\theta)\frac{[P_g(X,G,\theta,\tilde\theta)]^+}{v_g(X,Y,G,L,\theta,\tilde\theta)}\left(A(\theta)+[\,X(\theta)C_1^{\mathrm T}(\theta)+G^{\mathrm T}(\theta)D_{12}^{\mathrm T}(\theta)\quad B_1(\theta)\,]\Upsilon_X^{-1}(\theta)\begin{bmatrix}C_1(\theta)\\0\end{bmatrix}\right)$$
$$+\left(A^{\mathrm T}(\theta)+[\,C_1^{\mathrm T}(\theta)\quad 0\,]\Upsilon_X^{-1}(\theta)\begin{bmatrix}C_1(\theta)X(\theta)+D_{12}(\theta)G(\theta)\\B_1^{\mathrm T}(\theta)\end{bmatrix}\right)\frac{[P_g(X,G,\theta,\tilde\theta)]^+}{v_g(X,Y,G,L,\theta,\tilde\theta)}\rho_i(\theta)$$
$$-\tilde\rho_i(\theta,\tilde\theta)\frac{[P_g(X,G,\theta,\tilde\theta)]^+}{v_g(X,Y,G,L,\theta,\tilde\theta)}-\rho_i(\theta)[\,I\quad 0\,]\frac{[R_g(X,Y,\theta)]^+}{v_g(X,Y,G,L,\theta,\tilde\theta)}\begin{bmatrix}I\\0\end{bmatrix}, \tag{A.7}$$

$$\partial_{Y_i}\{v_g\} = \rho_i(\theta)\frac{[Q_g(Y,L,\theta,\tilde\theta)]^+}{v_g(X,Y,G,L,\theta,\tilde\theta)}\left(A^{\mathrm T}(\theta)+[\,Y(\theta)B_1(\theta)+L(\theta)D_{21}(\theta)\quad C_1^{\mathrm T}(\theta)\,]\Upsilon_Y^{-1}(\theta)\begin{bmatrix}B_1^{\mathrm T}(\theta)\\0\end{bmatrix}\right)$$
$$+\left(A(\theta)+[\,B_1(\theta)\quad 0\,]\Upsilon_Y^{-1}(\theta)\begin{bmatrix}B_1^{\mathrm T}(\theta)Y(\theta)+D_{21}^{\mathrm T}(\theta)L^{\mathrm T}(\theta)\\C_1(\theta)\end{bmatrix}\right)\frac{[Q_g(Y,L,\theta,\tilde\theta)]^+}{v_g(X,Y,G,L,\theta,\tilde\theta)}\rho_i(\theta)$$
$$+\tilde\rho_i(\theta,\tilde\theta)\frac{[Q_g(Y,L,\theta,\tilde\theta)]^+}{v_g(X,Y,G,L,\theta,\tilde\theta)}-\rho_i(\theta)[\,0\quad I\,]\frac{[R_g(X,Y,\theta)]^+}{v_g(X,Y,G,L,\theta,\tilde\theta)}\begin{bmatrix}0\\I\end{bmatrix}, \tag{A.8}$$

$$\partial_{G_i}\{v_g\} = 2\rho_i(\theta)\left(B_2^{\mathrm T}(\theta)+[\,D_{12}^{\mathrm T}(\theta)\quad 0\,]\Upsilon_X^{-1}(\theta)\begin{bmatrix}C_1(\theta)X(\theta)+D_{12}(\theta)G(\theta)\\B_1^{\mathrm T}(\theta)\end{bmatrix}\right)\frac{[P_g(X,G,\theta,\tilde\theta)]^+}{v_g(X,Y,G,L,\theta,\tilde\theta)}, \tag{A.9}$$

$$\partial_{L_i}\{v_g\} = \rho_i(\theta)\frac{[Q_g(Y,L,\theta,\tilde\theta)]^+}{v_g(X,Y,G,L,\theta,\tilde\theta)}\left(C_1^{\mathrm T}(\theta)+[\,Y(\theta)B_1(\theta)+L(\theta)D_{21}(\theta)\quad C_1^{\mathrm T}(\theta)\,]\Upsilon_Y^{-1}(\theta)\begin{bmatrix}D_{21}^{\mathrm T}(\theta)\\0\end{bmatrix}\right). \tag{A.10}$$

Set initial matrices $X^0 = (X^0)^{\mathrm{T}}$, $Y^0 = (Y^0)^{\mathrm{T}}$, G^0 *and* L^0.

At each step k, *generate* θ^k *in the set* Θ *according to the pdf* $f_\theta(\theta)$ *and* $\tilde{\theta}^k$ *in the set* $\tilde{\Theta}$ *according to the pdf* $f_{\tilde{\theta}}(\tilde{\theta})$ *and, for* $i = 1, \ldots, n_p$, *compute*

$$X_i^{k+1} = X_i^k - \mu^k \frac{\partial_{X_i}\{v_g(X^k, Y^k, G^k, L^k, \theta^k, \tilde{\theta}^k)\}}{w_g(X^k, Y^k, G^k, L^k, \theta^k, \tilde{\theta}^k)};$$

$$Y_i^{k+1} = Y_i^k - \mu^k \frac{\partial_{Y_i}\{v_g(X^k, Y^k, G^k, L^k, \theta^k, \tilde{\theta}^k)\}}{w_g(X^k, Y^k, G^k, L^k, \theta^k, \tilde{\theta}^k)};$$

$$G_i^{k+1} = G_i^k - \mu^k \frac{\partial_{G_i}\{v_g(X^k, Y^k, G^k, L^k, \theta^k, \tilde{\theta}^k)\}}{w_g(X^k, Y^k, G^k, L^k, \theta^k, \tilde{\theta}^k)};$$

$$L_i^{k+1} = L_i^k - \mu^k \frac{\partial_{L_i}\{v_g(X^k, Y^k, G^k, L^k, \theta^k, \tilde{\theta}^k)\}}{w_g(X^k, Y^k, G^k, L^k, \theta^k, \tilde{\theta}^k)}$$

if $v_g(X^k, Y^k, \theta^k, \tilde{\theta}^k) > 0$, *or*

$$X^{k+1} = X^k, \quad Y^{k+1} = Y^k, \quad G^{k+1} = G^k, \quad L^{k+1} = L^k,$$

if $v_g(X^k, Y^k, G^k, L^k, \theta^k, \tilde{\theta}^k) = 0$. *The stepsize* μ^k *is given by*

$$\mu^k \doteq \frac{v_g(X^k, Y^k, \theta^k, \tilde{\theta}^k)}{w_g(X^k, Y^k, \theta^k, \tilde{\theta}^k)} + r,$$

where the scalar parameter $r > 0$ *is the radius of a ball* $B_r \subset S^{\varepsilon}_{\Theta,\tilde{\Theta}}$.

Remark A.2. In the above algorithm, we can utilize a projection for the iterations of symmetric variables X^k and Y^k as in Remark 2, which may improve the convergence. However, such a projection cannot be introduced for nonsymmetric variables G^k and L^k. In this sense, the iteration formula without projection is crucial.

In the next theorem, the convergence properties of Algorithm 2 are stated.

Theorem A.1. *Let Assumptions* 3 *and* 4 *be satisfied. Then, Algorithm* 2 *converges with probability one in a finite number of iterations. That is,*

$$\mathrm{Prob}\{\exists k_0 < \infty: \\ (X^k, Y^k, G^k, L^k) \in S^{\varepsilon}_{\Theta,\tilde{\Theta}} \subset S^{0}_{\Theta,\tilde{\Theta}} \quad \forall k \geqslant k_0\} = 1,$$

where $S^{0}_{\Theta,\tilde{\Theta}}$ *is defined as the feasible solution set of Lemma* 7.

The above theorem is the exact counterpart of Theorem 4 for the regularized case. Hence, all the discussions reported after Theorem 4 still hold here. In particular, it can be shown that Algorithm 2 enjoys the same properties of boundedness and convergence of the solution stated in Section 3.3.

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