Hard Bounds on the Probability of Performance With Application to Circuit Analysis

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***Abstract*—In this paper, we address the problem of analyzing the performance of an electrical circuit in the presence of uncertainty in the network components. In particular, we consider the case when the uncertainties are known to be bounded and have probabilistic nature, and aim at evaluating the probability that a given system property holds. In contrast with the standard Monte Carlo approach, which utilizes random samples of the uncertainty to estimate "soft" bounds on this probability, we present a methodology that provides "hard" (deterministic) upper and lower bounds. To this aim, we develop an iterative algorithm, based on a *property oracle*, which is shown to converge asymptotically to the true probability of property satisfaction. Construction of the property oracles for specific applications in circuit analysis is explicitly presented. In particular, we study in full detail the problems of assessing the probability that the gain of a purely resistive network does not exceed a prescribed value, and of evaluating the probability of stability of an uncertain network under parameter variations. The paper is accompanied by illustrating examples and extensive numerical simulations.**

***Index Terms*—Circuit simulation, control theory, ladder networks, probabilistic methods, randomized algorithms, resistive circuits, robustness.**

I. Introduction

ANALYSIS of the performance of electrical circuits cannot disregard the unavoidable presence of uncertainty in the network components. This uncertainty usually arises from tolerances introduced by the manufacturing process and/or, in the case of thin-film circuits, imprecisions in the deposition processes.

Two different philosophies have been adopted in the literature to deal with simulation and analysis of uncertain networks. In the classical *statistical approach* used in circuit analysis, one assumes a stochastic description of the uncertainty, and aims at estimating the average – or expected – behavior of the circuit, see, e.g., [9], [11], [19], and [20]. Conversely, the *robust paradigm*, which emerged in the eighties in the systems and control community, does not assume any *a priori* knowledge on the statistical nature of the uncertainty but utilizes deterministic bounds on physical parameters to compute the system's worst-case performance, see [2] and references therein.

Manuscript received November 27, 2007; revised March 19, 2008. First published April 18, 2008; current version published November 21, 2008. This work was supported in part by the Consiglio Nazionale delle Ricerche (CNR) under the Short Term Mobility Program and the RSTL grant, and by the National Science Foundation (NSF) by Grants CNS-0519897 and ECCS-0501166.

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Digital Object Identifier 10.1109/TCSI.2008.923436

In recent years, in the context of robustness of uncertain systems, a novel probabilistic approach has been proposed for building a bridge between these two paradigms; see, for instance, [7], [21], and references therein. This approach has been mainly motivated by some pessimistic results on the complexity-theoretic barriers of deterministic robustness problems [4]. To overcome these limitations, the concept of *probability of performance* has been introduced and studied. Namely, consider a generic system $\mathcal{S}(q)$ which depends on a vector of uncertain parameters $q \in \mathbb{R}^\ell$. In the probabilistic approach, this uncertainty is assumed to be both bounded (in a hyperrectangle $\mathcal{Q}$) and random (with a given distribution). Then, we are interested in evaluating the probability that a given *system property* (for example, stability or performance), holds.

It should be noted that moving from a deterministic to a probabilistic paradigm does not automatically imply a simplification of the problem. Indeed, assessing probabilistic satisfaction of a given property may be computationally harder than assessing robustness in the usual deterministic sense. For these reasons, emphasis has been placed on the construction of algorithms for estimating such probabilities based on uncertainty randomization. Randomized algorithms provide *soft* estimates of the probability of performance using random samples: Since the estimated probability is itself a random quantity, this method always entails a certain *risk of failure*, i.e., there exists a nonzero probability of making an erroneous estimation. Tail probability inequalities are then used to bound the error of the estimate and the risk of failure. However, there are many situations in which the risk associated with these randomized techniques may be too high and practically not acceptable.

For this reason, in this paper we study a different approach to this problem, and we provide an algorithm for computing *hard bounds* on the probability of performance. The algorithm has sequential nature and recursively branches the uncertainty set, generating asymptotically converging upper and lower bounds on the probability of performance. These bounds are determined without resorting to randomization. Instead, they are based on sufficient conditions for either robust satisfaction of the considered system property, or its robust violation.

Before further elaborating on these concepts, we propose two motivating examples illustrating in detail the philosophy of this paper.

A. Example: Gain of a Resistive Network

Consider a planar resistive network, consisting of an input voltage source $V_{\rm in}$ and output voltage $V_{\rm out}$ across a designated

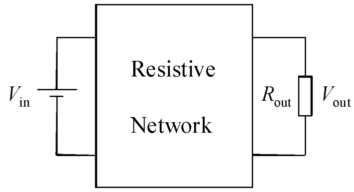

Fig. 1. Resistive network of Example I-A.

resistor $R_{\rm out}$, as depicted in Fig. 1. The gain of the resistive network is defined as the ratio of the output and input voltages

$$g \doteq \frac{V_{\rm out}}{V_{\rm in}}.$$

Suppose now that the network is formed by resistors $R_1, \ldots, R_\ell$ which are not perfectly known, but are instead subject to given tolerances as, e.g., in [8]. For instance, we may consider the case when the ith resistor has nominal manufacturing value $\bar{R}_i$, and is subject to a 10% manufacturing tolerance.

We pose the following question: *What is the probability that the gain of the circuit exceeds a prescribed value* γ*?* The value γ could, for instance, represent the gain when every resistor is set to its nominal value, or a safety threshold below which the system is guaranteed to perform correctly.

To be more formal, we let q_i denote the (uncertain) value of the ith resistor, and assume that the uncertainty vector

$$q \doteq [\,q_1 \quad \cdots \quad q_\ell\,]^\top \tag{1}$$

has a given probability distribution, say uniform, over the set

$$\mathcal{Q} \doteq \{q \in \mathbb{R}^\ell : q_i \in [q_i^-, q_i^+],\, i = 1, \ldots, \ell\}$$

(i.e., $q_i^- = 0.9\,\bar{R}_i$, $q_i^+ = 1.1\,\bar{R}_i$) and we want to evaluate the probability

$$\mathrm{P}\textsc{r}\{q \in \mathcal{Q} : g(q) \leq \gamma\}. \tag{2}$$

This probabilistic setup in the analysis and simulations of circuits is not new and has been formalized in [17], where precise guidelines are given for the choice of the uncertainty distribution.

B. Example: Stability of an Active Network

Stability in the context of circuits has been studied for instance in [13]. In this case, we consider a network similar to the one in Fig. 1, but containing both active and passive components and driven by an ac signal. Let the transfer function from the input voltage $V_{\rm in}(s)$ to the output voltage $V_{\rm out}(s)$ be given by

$$\frac{V_{\rm out}(s)}{V_{\rm in}(s)} = \frac{n(s,q)}{p(s,q)} \tag{3}$$

where $q \in \mathcal{Q}$ represents again uncertainties due to tolerances on the network components.

Suppose that the nominal system is stable, i.e., $p(s,q)$ is Hurwitz (all its roots lie in the open left half plane) when the q_i's are set to their nominal value. We are interested in analyzing the possibility that the circuit looses stability due to variations of its components around their manufacturing values. In other words, we are posing the following question: *What is the probability that the system remains stable?*

Again, this may be formally stated as the problem of evaluating the probability

$$\mathrm{P}\textsc{r}\{q \in \mathcal{Q} : p(s,q) \text{ is Hurwitz}\}.$$

NOTATION

A set $Q \subset \mathbb{R}^\ell$ is said to be a hyperrectangle if it is of the form

$$Q = \{q \in \mathbb{R}^\ell : q_i \in [q_i^-, q_i^+],\, i = 1, \ldots, \ell\}. \tag{4}$$

For a given hyperrectangle $Q \subset \mathbb{R}^\ell$, we define as $\mathrm{size}(Q)$ the maximum length of the edges of Q, that is

$$\mathrm{size}(Q) \doteq \max_{i=1,\ldots,\ell} (q_i^+ - q_i^-).$$

Moreover, given a set $\mathcal{A}$, we denote by $\mathrm{conv}(\mathcal{A})$ the convex hull of the set $\mathcal{A}$, by $\mathrm{cl}(\mathcal{A})$ its closure and by $\mathrm{vol}(\mathcal{A})$ its Lebesgue measure (volume).

II. PROBLEM FORMULATION

In this section, we present a general formulation that encompasses in a unique framework the previous illustrating examples and also many other situations arising in circuit analysis. To this end, given a generic *uncertain system* $\mathcal{S}(q)$, we consider a specific *system property* $\mathcal{P}$ and define the two sets

$$\begin{aligned} &\mathcal{Q}_{\rm good} \doteq \{q \in \mathcal{Q} : \mathcal{S}(q) \text{ satisfies Property } \mathcal{P}\} \\ \text{and} \quad &\mathcal{Q}_{\rm bad} \doteq \{q \in \mathcal{Q} : \mathcal{S}(q) \text{ does not satisfy Property } \mathcal{P}\}. \end{aligned}$$

We say that Property $\mathcal{P}$ is *well-defined* if the two sets above are Lebesgue measurable, see [14, Ch. 1,2]. This requirement is very mild and automatically satisfied by most "reasonable" system properties that one usually encounters in circuit analysis.

The uncertainty vector q is assumed to be random and uniformly distributed in a hyperrectangle $\mathcal{Q}$ of the form (4). We remark that the choice of uniform distribution is justified by its worst-case properties, see [1], [3], [17]. Moreover, it permits the interpretation of the probabilistic statements in terms of volumes of the good and bad sets. Further comments in this regard are made in the conclusions. Finally, we assume that the set $\mathcal{Q}$ has nonempty interior, that is $\mathrm{vol}(\mathcal{Q}) > 0$.

Given the definitions above, the problem under consideration is the following: Evaluate the probability that system $\mathcal{S}(q)$ satisfies Property $\mathcal{P}$; i.e., evaluate the *probability of performance*

$$\mathrm{P}_\mathcal{P} \doteq \mathrm{P}\textsc{r}\{\mathcal{Q}_{\rm good}\}.$$

A. Randomized Algorithms for Estimation of $\mathrm{P}_\mathcal{P}$

It is well known that probability $\mathrm{P}_\mathcal{P}$ can be estimated by means of a straightforward Monte Carlo technique. To this end, one extracts N independent identically distributed (i.i.d.) uniform samples $q^{(1)}, \dots, q^{(N)}$ of q and computes the so-called *empirical probability*

$$\widehat{\mathrm{P}}_{\mathcal{P}} = \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}(q^{(i)} \in \mathcal{Q}_{\mathrm{good}})$$

where the indicator function $\mathbb{I}(\cdot)$ is one if the clause is true, and it is zero otherwise. From elementary probability, we know that $\widehat{\mathrm{P}}_{\mathcal{P}} \to \mathrm{P}_{\mathcal{P}}$ as the number N of samples goes to infinity. However, in practice, it is important to know *a priori* how accurate is the estimate $\widehat{\mathrm{P}}_{\mathcal{P}}$ of $\mathrm{P}_{\mathcal{P}}$ when a *finite* and given number of samples is employed. Such an assessment is provided by the Hoeffding inequality [16], [21], which states that for given $\epsilon > 0$

$$\mathrm{P}\mathrm{R}_{\infty} \left\{ \left| \widehat{\mathrm{P}}_{\mathcal{P}} - \mathrm{P}_{\mathcal{P}} \right| \geq \epsilon \right\} \leq 2\mathrm{e}^{-2N\epsilon^2} \tag{5}$$

where the probability PR_{∞} is measured in the space of sample sequences $q^{(1)}, q^{(2)}, \dots$

Hence, if we set *a priori* the accuracy $\epsilon \in (0,1)$ and confidence level $\delta \in (0,1)$, i.e., if we set

$$2\mathrm{e}^{-2N\epsilon^2} \leq \delta,$$

then we obtain the so-called Chernoff bound for the sample complexity

$$N \geq \frac{1}{2\epsilon^2} \ln \frac{2}{\delta}. \tag{6}$$

This means that if the number of samples used for estimation satisfies (6), then we guarantee that

$$\mathrm{P}\mathrm{R}_{\infty} \left\{ \left| \widehat{\mathrm{P}}_{\mathcal{P}} - \mathrm{P}_{\mathcal{P}} \right| \leq \epsilon \right\} \geq 1 - \delta \tag{7}$$

that is, $\widehat{\mathrm{P}}_{\mathcal{P}}$ will be ϵ-close to $\mathrm{P}_{\mathcal{P}}$ with probability at least $1-\delta$.

This two-level probability assessment is unavoidable when using a randomized approach to estimate probability of performance. Bounds such as (7) are referred to as *soft bounds*, because they are not exact, but only guaranteed with probability $(1-\delta)$.

III. Hard Bounds on $\mathrm{P}_{\mathcal{P}}$ – The Oracle

As aforementioned, in many practical applications, such as those discussed in the motivating examples, soft bounds may not be satisfactory. For this reason, in this paper we present an algorithm that provides hard (i.e., deterministic) lower and upper bounds on $\mathrm{P}_{\mathcal{P}}$. The algorithm we propose is based on the existence of an oracle, as discussed next.

We assume that an oracle $\mathcal{O}$ is available for testing if, for any given hyperrectangle $Q \subseteq \mathcal{Q}$, the Property $\mathcal{P}$ is satisfied by $\mathcal{S}(q)$ for all $q \in Q$. The oracle does not have to be exact: It may leave some points undecided, at least for a while.

Definition 1 (Oracle): Given a hyperrectangle $Q \in \mathcal{Q}$, define the following oracle $\mathcal{O}$:

i) $\mathcal{O}(Q) = \mathrm{TRUE}$ *implies that Property $\mathcal{P}$ is satisfied by $\mathcal{S}(q)$ for all $q \in Q$.*

ii) $\mathcal{O}(Q) = \mathrm{FALSE}$ *implies that Property $\mathcal{P}$ is not satisfied by $\mathcal{S}(q)$ for all $q \in Q$.*

iii) $\mathcal{O}(Q) = \mathrm{UNDECIDED}$ *if cases i) and ii) could not be proven by the oracle.* $\diamondsuit$

The oracle $\mathcal{O}$ is said to be a $\mathcal{P}$–Oracle if the set of undecidable points $\mathcal{U}$ (points for which the oracle cannot determine if the Property $\mathcal{P}$ holds or not) has volume zero. In other words, if $\mathcal{O}(Q) = \mathrm{UNDECIDED}$ then for every $\tilde{q} \in Q$ that does not belong to the undecidable set $\mathcal{U}$ the oracle will eventually provide the right answer. Formally, the $\mathcal{P}$-Oracle should comply with the following definition.

Definition 2 ($\mathcal{P}$–Oracle): The oracle $\mathcal{O}$ is said to be a $\mathcal{P}$–Oracle if, for any $\tilde{q} \in Q \setminus \mathcal{U}$ there exists $\varepsilon > 0$ such that for any hyperrectangle $\tilde{Q}$ satisfying $\tilde{q} \in \tilde{Q}$ and $\mathrm{size}(\tilde{Q}) \leq \varepsilon$ one of the following two implications hold:

$$\tilde{q} \text{ satisfies Property } \mathcal{P} \Rightarrow \mathcal{O}(\tilde{Q}) = \mathrm{TRUE};$$
$$\tilde{q} \text{ does not satisfy Property } \mathcal{P} \Rightarrow \mathcal{O}(\tilde{Q}) = \mathrm{FALSE}. \qquad \diamondsuit$$

Remark 1 (Oracle Misclassification): Notice that the definition above allows the $\mathcal{P}$–Oracle to misclassify a hyperrectangle Q. In particular, the oracle may return UNDECIDED when Property $\mathcal{P}$ is satisfied (unsatisfied) for all $q \in Q$. In other words, the oracle does not have to be exact. However, we do not allow the oracle to misclassify a TRUE or FALSE answer: If the oracle returns TRUE then $\mathcal{S}(q)$ satisfies Property $\mathcal{P}$ for all $q \in Q$. If the oracle returns FALSE then $\mathcal{S}(q)$ does not satisfy Property $\mathcal{P}$ for all $q \in Q$. $\star$

The proposed algorithm is presented in the following section, under the assumption that a $\mathcal{P}$–Oracle complying with Definition 2 is available. Examples of $\mathcal{P}$-Oracles for different circuit applications are given in Sections V and VI.

IV. Hard Bounds on $\mathrm{P}_{\mathcal{P}}$ – The Algorithm

Given a $\mathcal{P}$–Oracle $\mathcal{O}$, we propose an iterative algorithm (Algorithm 1) for computing upper and lower bounds on the probability of satisfaction of Property $\mathcal{P}$. The algorithm iteratively updates a list $\mathcal{L}$ containing subsets of $\mathcal{Q}$, and it is based on $\mathcal{P}$-oracle calls. If an element of $\mathcal{L}$ is labeled TRUE (FALSE) then it is deemed to be a subset of $\mathcal{Q}_{\mathrm{good}}$ ($\mathcal{Q}_{\mathrm{bad}}$). Otherwise, if the oracle returns an UNDECIDED answer, the set is split into two parts along its longest edge, and the two subsets are added at the end of the list $\mathcal{L}$.

Algorithm 1 Computes hard probability bounds $\mathrm{P}_{\mathcal{P}}^{+}$ and $\mathrm{P}_{\mathcal{P}}^{-}$ such that $\mathrm{P}_{\mathcal{P}}^{-} \leq \mathrm{P}_{\mathcal{P}} \leq \mathrm{P}_{\mathcal{P}}^{+}$ and $\mathrm{P}_{\mathcal{P}}^{+} - \mathrm{P}_{\mathcal{P}}^{-} \leq \epsilon$.

Require: $\epsilon > 0$, $\mathcal{P}$-Oracle

Ensure: $\mathrm{P}_{\mathcal{P}}^{+}$ and $\mathrm{P}_{\mathcal{P}}^{-}$

INITIALIZATION

$k \Leftarrow 0, \mathcal{L} \Leftarrow \{\mathcal{Q}\}, \mathrm{P}_{\mathcal{P}}^{+}(k) \Leftarrow 1, \mathrm{P}_{\mathcal{P}}^{-}(k) \Leftarrow 0$

HYPERRECTANGLE SELECTION

remove the first element of $\mathcal{L}$ and name it Q^k.

$\mathcal{P}$-*ORACLE*

if $\mathcal{O}(Q^k) = \text{TRUE}$ **then**

$\mathrm{P}_{\mathcal{P}}^{-}(k+1) = \mathrm{P}_{\mathcal{P}}^{-}(k) + \mathrm{vol}(Q^k)/\mathrm{vol}(\mathcal{Q})$

else if $\mathcal{O}(Q^k) = \text{FALSE}$ **then**

$\mathrm{P}_{\mathcal{P}}^{+}(k+1) = \mathrm{P}_{\mathcal{P}}^{+}(k) - \mathrm{vol}(Q^k)/\mathrm{vol}(\mathcal{Q})$

else if $\mathcal{O}(Q^k) = \text{UNDECIDED}$ **then**

split Q^k along the longest edge, and put at the end of $\mathcal{L}$ the two hyperrectangles obtained.

end if

EXIT CONDITION

if $\mathrm{P}_{\mathcal{P}}^{+}(k) - \mathrm{P}_{\mathcal{P}}^{-}(k) \leq \epsilon$ **then**

return $\mathrm{P}_{\mathcal{P}}^{+}(k)$, $\mathrm{P}_{\mathcal{P}}^{-}(k)$

else

let $k = k + 1$ and goto 2

end if

The upper and lower bounds provided by Algorithm 1, by construction, have the following property for all k:

$$\mathrm{P}_{\mathcal{P}}^{+}(k+1) \leq \mathrm{P}_{\mathcal{P}}^{+}(k); \qquad \mathrm{P}_{\mathcal{P}}^{-}(k+1) \geq \mathrm{P}_{\mathcal{P}}^{-}(k).$$

Furthermore, the bounds are asymptotically converging, as stated in the following theorem, see Appendix A for a formal proof.

Theorem 1: Given Property $\mathcal{P}$, assume that a $\mathcal{P}$–Oracle is available (satisfying the conditions of Definition 2). Then

$$\lim_{k\to\infty} \mathrm{P}_{\mathcal{P}}^{+}(k) = \lim_{k\to\infty} \mathrm{P}_{\mathcal{P}}^{-}(k) = \mathrm{P}_{\mathcal{P}}.$$

Remark 2 (Complexity): One should note that, when computing deterministic estimates of volume (or probability), one is faced with a problem that is known to be NP–Hard [12]. Hence, it is not surprising that the computational complexity of the proposed algorithm increases exponentially with the number of uncertain parameters. The need of considering an exponential-time algorithm in order to have a single level of probability was already discussed in [1]. However, the upper and lower bounds obtained are always *hard bounds*; i.e., independently of when the algorithm is stopped, the probability of stability is guaranteed to lie in the interval defined by these bounds. $\star$

As seen in Theorem 1, a crucial requirement for the application of Algorithm 1 is the availability of a specific property oracle. For this reason, in the sequel of this paper we revisit the motivating examples discussed in the Introduction and, for each of them, we explicitly show how a specific $\mathcal{P}$-Oracle can be constructed.

V. PROBABILITY OF EXCEEDING A GIVEN GAIN

As a first application of the proposed methodology, we revisit Example I-A, and study the problem of evaluating the probability that the gain of a resistive network remains below a prescribed threshold. Formally, we define the uncertain gain

$$g(q) \doteq \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{n(q)}{p(q)} \tag{8}$$

where q is the uncertainty vector (1), i.e., q_i represents the numerical value of the ith resistor. We assume that the denominator $p(q)$ is nonzero for all values of the uncertainty, that is $p(q) \neq 0$ for all $q \in \mathcal{Q}$. This assumption is automatically satisfied when all resistors have strictly positive range of variation. Then, for a given threshold level γ, we are interested in evaluating the probability of the set

$$\mathcal{Q}_{\text{good}} \doteq \{q \in \mathcal{Q} : g(q) \leq \gamma\}.$$

Lemma A in [17] proves that the gain $g(q)$ of a resistive network is the ratio of two *multiaffine*[1] functions of q, that is $n(q)$ and $p(q)$ are multiaffine in q. Hence, from [6, Theorem 1.4], it follows that the maximum and minimum $\overline{g}(Q)$ and $\underline{g}(Q)$ of $g(q)$ are attained at a vertex of the set $\mathcal{Q}$. The following lemma shows how to construct a $\mathcal{P}$–Oracle satisfying the conditions of Definition 2.

Lemma 1 (Oracle for Worst-Case Gain): Given a hyperrectangle $Q \subseteq \mathcal{Q}$, compute $\overline{g}(Q)$ and $\underline{g}(Q)$. Then, a $\mathcal{P}$–Oracle satisfying the conditions of Definition 2 is the following:

$$\mathcal{O}(Q) = \begin{cases} \text{FALSE} & \text{if } \underline{g}(Q) > \gamma; \\ \text{TRUE} & \text{if } \overline{g}(Q) \leq \gamma; \\ \text{UNDECIDED} & \text{otherwise.} \end{cases}$$

Proof: We have already shown that the gain is attained at a vertex of the set. All we have to prove is that the set of undecided points has measure zero. This is an immediate consequence of the fact that the set $\mathcal{U}$ is, in this case, the boundary of the set $\mathcal{Q}_{\text{good}}$. Since $\mathcal{Q}_{\text{good}}$ is a Lebesgue measurable set then its boundary has measure zero. $\square$

VI. PROBABILITY OF STABILITY OF AN ACTIVE NETWORK

The second application we discuss is related to Example I-B, and concerns the evaluation of the probability of stability of an uncertain polynomial.

In this case, the uncertain system $\mathcal{S}(q)$ is given by the uncertain polynomial $p(s,q)$ (e.g., the denominator of the transfer function in (3)), that we assume to be monic, i.e.

$$p(s,q) = a_0(q) + a_1(q)s + \cdots + a_{n-1}(q)s^{n-1} + s^n. \tag{9}$$

The uncertain parameter vector $q \in \mathbb{R}^\ell$ is a random vector uniformly distributed over the hyperrectangle $\mathcal{Q}$. We consider here the important case where the coefficients $a_i(q), i = 1, \ldots, n-1$ are multiaffine functions of the uncertain parameters q_i's. We are interested in evaluating the probability of the polynomial $p(s,q)$ being Hurwitz; that is we aim at providing hard bounds on the probability

$$\mathrm{P}_{\mathcal{P}} \doteq \mathrm{PR}\{\mathcal{Q}_{\text{good}}\}$$

where $\mathcal{Q}_{\text{good}} = \{q \in \mathcal{Q} : p(s,q) \text{ is Hurwitz}\}$.

[1]A function $f : \mathbb{R}^\ell \to \mathbb{R}$ is said to be multiaffine if the following condition holds: If all components $q_1, \ldots, q_\ell$ except one are fixed, then f is affine. For example, $f(q) = 3q_1q_2q_3 - 6q_1q_3 + 4q_2q_3 + 2q_1 - 2q_2 + q_3 - 1$ is multiaffine.

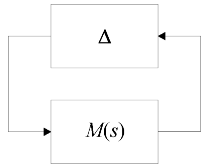

Fig. 2. M-Δ uncertainty model.

Remark 3 (Generality of the Setup): We remark that the proposed framework is quite general and can be utilized in many robustness analysis problems. In particular, the setup encompasses the case when the uncertain system under study is expressed in the standard M-Δ configuration of Fig. 2, see for instance [22, Ch. 9]. In fact, letting

$$\Delta = \mathrm{diag}\{q_1, q_2, \ldots, q_\ell\}, \quad q \in \mathcal{Q}$$

and assuming $M(s)$ stable, we have that the M-Δ interconnection in Fig. 2 is robustly stable if and only

$$\det\left(I - M(s)\Delta\right) \neq 0 \quad \forall q \in \mathcal{Q}, \quad \forall s = j\omega, \, \omega \in \mathbb{R}.$$

Notice that, from Lemma 1 in [15], the numerator of

$$\det\left(I - M(s)\Delta\right),$$

which we denote by $p(s,q)$, is a multiaffine polynomial[2] in q. At this point, since $M(s)$ is stable, we can use a zero exclusion reasoning (see, e.g., [2, Ch. 5.7.8]) and notice that $\det\left(I - M(s)\Delta\right)$ vanishes if and only if $p(s,q)$ is unstable for some $q \in \mathcal{Q}$. Therefore, robust stability of the M-Δ configuration is equivalent to robust stability of the multiaffine polynomial $p(s,q)$ with $q \in \mathcal{Q}$.

An approach similar to the one followed in this section has been proposed in [23] for computing a hard upper bound on the probability of the largest structured singular value $\mu(M(j\omega))$ being less than a given level γ. However, it is not clear how this probability, computed at fixed frequency, can be related to the probability of robust stability of the M-Δ configuration, which is the one of interest. ⋆

In the next section, we consider the problem of constructing a $\mathcal{P}$–Oracle satisfying the requirements of Definition 2 for the general case of polynomials with multiaffine uncertainty, while in Section VI-B we specialize our results to the case of $p(s,q)$ being an interval polynomial.

A. An Oracle for Multiaffine Polynomial

We first need to introduce an additional concept which is instrumental to the oracle definition: For a given frequency $\omega \in \mathbb{R}$, we define the *value set* of the set Q as the following subset of the complex plane:

$$\mathcal{V}(Q,\omega) \doteq \{p(j\omega, q) : \, q \in Q\}.$$

The following lemma provides an oracle for testing Hurwitz stability of a given hyperrectangle, in the case of polynomials with multiaffine uncertainty: The idea at the basis of the oracle is a test of inertia invariance[3] of the roots of the polynomial $p(s,q)$.

Lemma 2 (Oracle for Stability of Multiaffine Polynomials): Assume that the set of $q \in \mathcal{Q}$ for which $p(s,q)$ has a root on the imaginary axis has measure zero. Then, a $\mathcal{P}$–Oracle satisfying the conditions of Definition 2 is the following: Let $Q \subseteq \mathcal{Q}$ be a hyperrectangle and $q_c(Q)$ its center. Then, (see the equation at the bottom of the page).

Proof: See Appendix B. □

Remark 4 (Complexity of Oracle Test): Notice that, from the Mapping Theorem (see, e.g., [2, Ch. 14.6]), the convex hull of the value set $\mathrm{conv}(\mathcal{V}(Q,\omega))$ can be constructed using the vertices of the hyperrectangle Q. Moreover, using the results in [10], we can limit the check to a finite number of "critical frequencies." ⋆

B. An Oracle for Interval Polynomials

In this section, we specialize the oracle previously presented to the case when $p(s,q)$ is an interval polynomial, i.e., the coefficients in (9) assume the simple form $a_i(q) = q_{i+1}$. In this case, the construction of an oracle satisfying Definition 2 is simpler.

To improve the numerical efficiency of the given oracle, we consider two cases separately: i) the "center" polynomial is stable and ii) the "center" polynomial is unstable. As discussed below, by treating these two cases separately, the performance of the oracle can be greatly improved. We begin the discussion by addressing the problem of testing zero exclusion for the case of interval polynomials. These results are instrumental to the definition of a $\mathcal{P}$–Oracle.

Note that, for interval polynomials, the value set is a rectangle with edges parallel to the real and imaginary axes and corners corresponding to the four Kharitonov polynomials, see for instance [2, Fig. 5.7.1]. This fact allows us to derive simple necessary and sufficient conditions for zero exclusion.

1) Case 1: Stable Polynomials: First, we recall that robust stability of an interval polynomial

$$p(s,q) = q_1 + q_2 s + q_3 s^2 + \cdots + q_\ell s^{\ell-1} + s^\ell$$

[2]Computed without performing pole/zero cancellations.

[3]The uncertain polynomial $p(s,q)$, $q \in Q$, is said to be inertia invariant if the number of roots in the open left-half plane does not change (is invariant) for all $q \in Q$.

$$\mathcal{O}(Q) = \begin{cases} \textsc{True} & \text{if } p(s, q_c(Q)) \text{ is stable and } 0 \notin \mathrm{conv}\left(\mathcal{V}(Q,\omega)\right), \forall \omega \in \mathbb{R}; \\ \textsc{False} & \text{if } p(s, q_c(Q)) \text{ is unstable and } 0 \notin \mathrm{conv}\left(\mathcal{V}(Q,\omega)\right), \forall \omega \in \mathbb{R}; \\ \textsc{Undecided} & \text{otherwise.} \end{cases}$$

with coefficients $q_i \in [q_i^-, q_i^+]$, is equivalent to the stability of the four Kharitonov polynomials [18], [5]

$$\begin{aligned} p_1(s) &\doteq q_1^- + q_2^- s + q_3^+ s^2 + q_4^+ s^3 + q_5^- s^4 + q_6^- s^5 + \cdots \\ p_2(s) &\doteq q_1^+ + q_2^+ s + q_3^- s^2 + q_4^- s^3 + q_5^+ s^4 + q_6^+ s^5 + \cdots \\ p_3(s) &\doteq q_1^+ + q_2^- s + q_3^- s^2 + q_4^+ s^3 + q_5^+ s^4 + q_6^- s^5 + \cdots \\ p_4(s) &\doteq q_1^- + q_2^+ s + q_3^+ s^2 + q_4^- s^3 + q_5^- s^4 + q_6^+ s^5 + \cdots. \end{aligned}$$

Hence, the zero exclusion test is greatly simplified whenever it is known that at least one of the members of the family is stable (for instance, the center). In fact, in this case, stability of the four Kharitonov polynomials guarantees that the value set never contains the origin.

2) Case 2: Unstable Polynomials: First we rewrite $p(s,q)$ in the form

$$p(s,q) = (q_1 + q_3 s^2 + \cdots) + s(q_2 + q_4 s^2 + \cdots) \doteq f(s^2) + s g(s^2).$$

Now, define

$$\begin{aligned} f^-(s^2) &\doteq q_1^- + q_3^+ s^2 + q_5^- s^4 + q_7^+ s^6 + \cdots \\ f^+(s^2) &\doteq q_1^+ + q_3^- s^2 + q_5^+ s^4 + q_7^- s^6 + \cdots \\ g^-(s^2) &\doteq q_2^- + q_4^+ s^2 + q_6^- s^4 + q_8^+ s^6 + \cdots \\ g^+(s^2) &\doteq q_2^+ + q_4^- s^2 + q_6^+ s^4 + q_8^- s^6 + \cdots. \end{aligned}$$

Hence, the four Kharitonov polynomials are given by

$$\begin{aligned} p_1(s) &= f^-(s^2) + s\, g^-(s^2) \\ p_2(s) &= f^+(s^2) + s\, g^+(s^2) \\ p_3(s) &= f^+(s^2) + s\, g^-(s^2) \\ p_4(s) &= f^-(s^2) + s\, g^+(s^2) \end{aligned}$$

and the corner polynomials take a particularly simple expression. These considerations lead us to the following lemma.

Lemma 3: Let $Q = \{q \in \mathbb{R}^\ell : q_i \in [q_i^-, q_i^+],\ i = 1, \ldots, \ell\}$ be a hyperrectangle. Assume without loss of generality that $q_1^- > 0$. Denote by $\Omega_p \doteq \{\omega_1^2, \omega_2^2, \ldots\}$ the set of all real positive distinct roots of the polynomial equation $p(-\omega^2) = 0$. Then, $0 \in \mathcal{V}(Q,\omega)$ if and only if the following conditions hold:

i) $f^-(-\omega_i^2) \le 0 \le f^+(-\omega_i^2),\ \forall \omega_i^2 \in \Omega_{g^-}$;
ii) $f^-(-\omega_i^2) \le 0 \le f^+(-\omega_i^2),\ \forall \omega_i^2 \in \Omega_{g^+}$;
iii) $g^-(-\omega_i^2) \le 0 \le g^+(-\omega_i^2),\ \forall \omega_i^2 \in \Omega_{f^-}$;
iv) $g^-(-\omega_i^2) \le 0 \le g^+(-\omega_i^2),\ \forall \omega_i^2 \in \Omega_{f^+}$.

Sketch of the Proof: This result is a consequence of the fact that, since $s = 0$ is not a root of $p(s,q)$ for some $q \in Q$, then $0 \in \mathcal{V}(Q,\omega)$ for some $\omega \in \mathbb{R}$ if and only if there exists $\omega^* \in \mathbb{R}$ such that 0 belongs to the boundary of $\mathcal{V}(Q,\omega^*)$. The conditions above are obtained by using the fact that the value set $\mathcal{V}(Q,\omega)$ is a rectangle for all $\omega \in \mathbb{R}$. □

Combining Lemma 2 and Lemma 3 we derive the following result, that provides a specific $\mathcal{P}$-Oracle for the case of interval polynomials.

Lemma 4 (Oracle for Stability of Interval Polynomials): Assume that the set of $q \in \mathcal{Q}$ for which $p(s,q)$ has a root on the imaginary axis has measure zero. Then, a $\mathcal{P}$–Oracle satisfying the conditions of Definition 2 is the following:

i) $p(s, q_c(Q))$ *is stable*: In this case, we have the following: (see the upper equation at the bottom of the page)
ii) $p(s, q_c(Q))$ *is unstable*: In this case, the oracle is defined as shown in the lower equation at the bottom of the page.

Remark 5 (Oracle Complexity for Interval Polynomials): Note that, testing zero exclusion can be performed with very low computational effort, requiring only a stability check of the four Kharitonov polynomials for a stable center polynomial, and a check of the values of four polynomials at the critical frequencies in the case of unstable center. ⋆

VII. Numerical Examples

In this section, we present some numerical examples that illustrate the behavior of the proposed algorithm for different circuit analysis problems.

A. Example: An Illustrative Two-Dimensional Case

We first consider an example of stability analysis with only two uncertain parameters. This enables us to provide a "visual" illustration of the behavior of the algorithm. Consider an interval polynomial

$$p(s,q) = s^6 + q_1 s^5 + 4s^4 + 6s^3 + q_2 s^2 + 2s + 0.5$$

where

$$q_1 \in [1,5] \text{ and } q_2 \in [2,4].$$

In Fig. 3, a plot of the partition of the uncertainty set performed by the algorithm is presented, where darker shade blue indicates the set of pairs (q_1, q_2) for which the polynomial $p(s,q)$ is stable. Lighter shade red indicates instability.

From this figure, one can see how the algorithm works. Since, in this case, an exact oracle for checking invariant inertia is available, the rectangles corresponding to a polynomial with invariant inertia can be directly classified either as robustly stable or robustly unstable. If the inertia inside the rectangle is not invariant, then the rectangle is partitioned and the oracle is applied to each of the partitions. This leads to rough partitioning

$$\mathcal{O}(Q) = \begin{cases} \textsc{True} & \text{if the Kharitonov polynomials associated with } Q \text{ are stable;} \\ \textsc{Undecided} & \text{otherwise.} \end{cases}$$

$$\mathcal{O}(Q) = \begin{cases} \textsc{Undecided} & \text{if the conditions of Lemma 3 are satisfied for the hyperrectangle } Q; \\ \textsc{False} & \text{otherwise.} \end{cases}$$

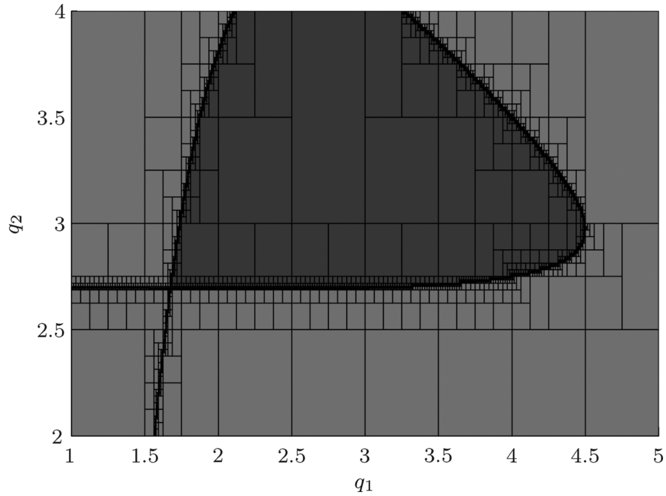

Fig. 3. Partition generated by the proposed algorithm for Example VII-A.

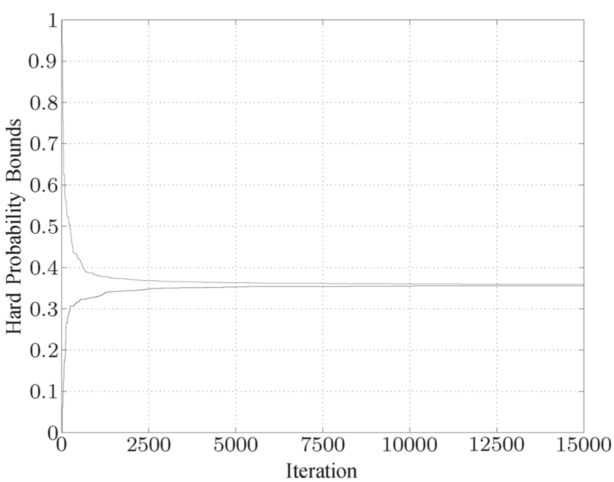

Fig. 4. Hard upper and lower bounds on the probability of stability for Example VII-A.

inside the inertia invariant regions and fine partitioning near the boundary of these regions. A plot of the upper and lower bounds on the probability of stability computed with Algorithm 1 is reported in Fig. 4.

For $\varepsilon = 0.05$, Algorithm 1 converged after 15 000 iterations to the upper and lower bounds

$$\mathrm{P}_{\mathcal{P}}^{+} = 0.3595 \text{ and } \mathrm{P}_{\mathcal{P}}^{-} = 0.3559$$

respectively. For comparison purposes, a probabilistic estimate has been computed using 15 000 random samples obtaining the randomized approximation $\widehat{\mathrm{P}}_{\mathcal{H}} = 0.3587$. Setting $\varepsilon = 0.05$ in the Hoeffding inequality (5) we see that this random estimate guarantees the accuracy

$$|\widehat{\mathrm{P}}_{\mathcal{H}} - \mathrm{P}_{\mathcal{H}}| \leq 0.05$$

with a probability of 99.9%.

B. Example: Resistive Ladder Network

We consider now the three-loop ladder network depicted in Fig. 5, which contains nine resistors. This configuration has

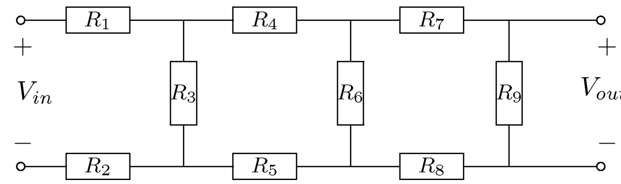

Fig. 5. Three-loop resistive ladder network.

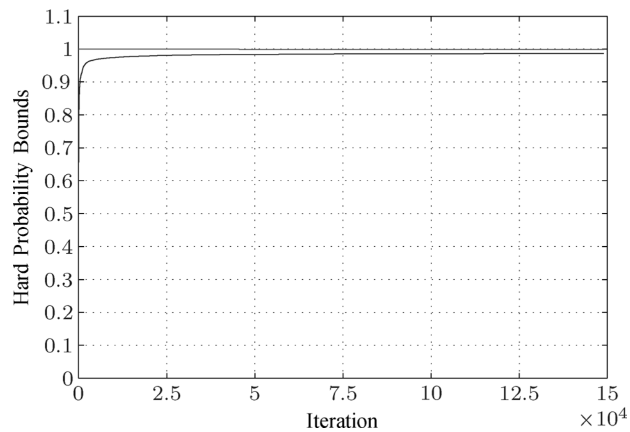

Fig. 6. Hard upper and lower bounds on the probability of stability for Example VII-B.

been subject of research in [17] within a probabilistic distributionally robust approach. The *gain* of the network, defined as the ratio between the output voltage V_{out} around resistor R_9 and input voltage V_{in}, can be computed using Kirchhoff rule as

$$g = \frac{R_3 R_6 R_9}{\det(M_R)}$$

where the resistance matrix of the network is given by

$$M_R = \begin{bmatrix} R_1+R_2+R_3 & -R_3 & 0 \\ -R_3 & R_3+R_4+R_5+R_6 & -R_6 \\ 0 & -R_6 & R_6+R_7+R_8+R_9 \end{bmatrix}.$$

We denote by q_i the numerical values of the ith resistor, and consider the same setup of [17, Example V.C]. That is, we assume that resistors R_3, R_6 and R_9 are fixed at their nominal values (3, 5, 7 Ohm, respectively), while the others are uniformly distributed in the intervals

$$\begin{aligned} &[0.9,\ 1.1], \quad \text{for } R_1, R_4, R_5, R_7 \text{ and } R_8 \\ &[1.8,\ 2.2], \quad \text{for } R_2. \end{aligned}$$

Due to parameter variations, we have that the actual gain $g(q)$ differs from the nominal $g_0 = g(0) = 0.1862$. Consider now the case when, for safety reasons, we need to guarantee that $g(q)$ exceeds the level $\gamma = 2$ with very low probability.

To evaluate hard bounds on the probability that the gain remains under the threshold γ, we constructed a $\mathcal{P}$-Oracle according to Lemma 1, and run Algorithm 1 setting $\epsilon = 0.01$.

The behavior of the hard bounds $\mathrm{P}_{\mathcal{P}}^{-}$ and $\mathrm{P}_{\mathcal{P}}^{+}$ is presented in Fig. 6. As expected, since in this case we have six uncertain pa-

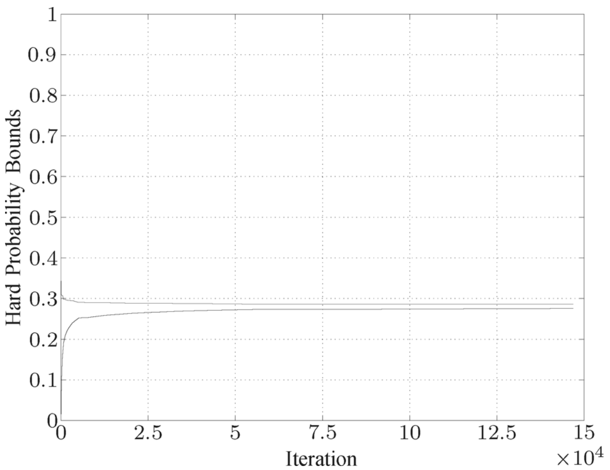

Fig. 7. Hard upper and lower bounds on the probability of stability for Example VII-C.

rameters, convergence is slower. However, after about 150 000 iterations, the bounds converged to the following values:

$$\mathrm{P}_{\mathcal{P}}^{+} = 0.9985; \qquad \mathrm{P}_{\mathcal{P}}^{-} = 0.9885$$

that are within 0.01.

C. Example: Stability of an Interval Polynomial

As a final example, we consider the interval polynomial

$$p(s, q) = (s+1)^3(s^2 + 0.002s + 1) + \sum_{i=1}^{5} q_i s^{i-1}$$

with interval coefficients

$$q_i \in [-0.03, 0.03]; \quad i = 1, 2, \ldots, 5.$$

Algorithm 1 was applied to this family of polynomials. The behavior of the hard bounds on the probability of stability is presented in Fig. 7. Also in this case, after about 150 000 iterations, the bounds converged to the following values:

$$\mathrm{P}_{\mathcal{P}}^{+} = 0.2858; \qquad \mathrm{P}_{\mathcal{P}}^{-} = 0.2758$$

that are within 0.01.

VIII. Conclusion and Future Directions

In this paper, we presented a new approach for computing deterministic upper and lower bounds on the probability of performance. The algorithm depends on the existence of a (nonexact) oracle. Specific $\mathcal{P}$-Oracles have been developed for different problems arising in uncertain circuit analysis. The proposed algorithm represents a valid and complementary analysis tool, to be applied whenever the degree of risk that comes along with randomized techniques is not considered sufficiently low.

Extensions of the presented methodology to densities other than uniform seem promising. In particular, the case when the uncertain parameters q are not uniformly distributed, but are still independent in the hyperrectangle $\mathcal{Q}$, falls in the framework of this work. Assume for instance that the probability density function of q can be written as $f_q(q) = \prod_{i=1}^{\ell} f_{q_i}(q_i)$, where $f_{q_i}(q_i)$ are univariate densities with support $[q_i^-, q_i^+]$. In this case, the ratio of volumes $\mathrm{vol}(Q^k)/\mathrm{vol}(\mathcal{Q})$ appearing in Algorithm 1 can be immediately substituted with the probability measure of the set Q^k, that is

$$\int_{Q^k} f_q(q)\mathrm{d}q = \prod_{i=1}^{\ell} \int_{\theta_i^-}^{\theta_i^+} f_{q_i}(q_i)\mathrm{d}q_i$$

where $[\theta_i^-, \theta_i^+]$ is the ith edge of the hyperrectangle Q^k.

Further effort is now being put in the optimization of the numerical performance of the algorithm. Moreover, the extension of the results to more general polynomial dependence on the uncertain parameters is currently under investigation.

Appendix A
Proof of Theorem 1

Let $\mathcal{L}(k) = \{Q_1(k), Q_2(k), \ldots, Q_{n_k}(k)\}$ be the hyperrectangles in the list $\mathcal{L}(k)$ at step k. In other words, these are the sets for which no decision has been made yet. Define further

$$S(k) \doteq \bigcup_{i=1}^{n_k} Q_i(k) = \bigcup_{Q \in \mathcal{L}^k} Q$$

which is the union of all undecided hyperrectangles at step k. We now prove that

$$\lim_{k\to\infty} \mathrm{vol}(S(k)) = 0 \tag{10}$$

which implies the desired result. This is the case, since if Q is not in the list, it means that it has been (correctly) classified as robustly satisfying the performance conditions or robustly violating it. Hence, if the volume of $S(k)$ converges to zero, both upper and lower bounds converge to the desired value.

We first state some properties of the hyperrectangles $Q_i(k)$, which can be easily derived from the definition of the oracle and the "mechanics" of the algorithm presented.

Fact 1: For all k and i we have

i) $\max_{i=1,2,\ldots,n_{k+1}} \mathrm{size}(Q_i(k+1)) \le \max_{i=1,2,\ldots,n_k} \mathrm{size}(Q_i(k))$.

ii) $\lim_{k\to\infty} \max_{i=1,2,\ldots,n_k} \mathrm{size}(Q_i(k)) = 0$.

Fact 2: Algorithm 1 implies that for all $k = 1, 2, \ldots$

i) *$S(k+1) \subseteq S(k)$ and hence $S(k) = \bigcap_{i=1}^{k} S(i)$.*

ii) $\mathrm{cl}(\mathcal{U}) \subseteq S(k)$.

Now let $q^* \notin \mathrm{cl}(\mathcal{U})$. Then, by Definition 2, there exists $\varepsilon^* > 0$ such that for any hyperrectangle Q satisfying

$$\mathrm{size}(Q) < \varepsilon^* \text{ and } \mathcal{O}(Q) = \text{UNDECIDED}$$

we have $q^* \notin Q$. In addition, by Fact 1, there exists k^* such that

$$\max_{i=1,2,\ldots,n_{k^*}} \mathrm{size}(Q_i(k^*)) < \varepsilon^*.$$

Hence, since $\mathcal{O}(Q_i(k^*)) = \text{UNDECIDED}$ for $i = 1, 2, \ldots, n_{k^*}$, we have

$$q^* \notin Q_i(k^*)$$

for $i = 1, 2, \ldots, n_{k^*}$ and, hence, $q^* \notin S(k^*)$. Therefore, if $q^* \notin \mathrm{cl}(\mathcal{U})$, then

$$q^* \notin \bigcap_{k=1}^{\infty} S(k).$$

Moreover, by Fact 2

$$\mathrm{cl}(\mathcal{U}) \subseteq \bigcap_{k=1}^{\infty} S(k).$$

Hence

$$\mathrm{cl}(\mathcal{U}) = \bigcap_{k=1}^{\infty} S(k).$$

To complete the proof, just note that standard results in measure theory (see for instance [14, Ch. 1,2]), together with Definition 2 and Fact 2, imply that

$$\begin{aligned}\lim_{k\to\infty} \mathrm{vol}(S(k)) &= \lim_{k\to\infty} \mathrm{vol}\left(\bigcap_{i=1}^{k} S(i)\right)\\ &= \mathrm{vol}\left(\bigcap_{i=1}^{\infty} S(i)\right) = \mathrm{vol}\left(\mathrm{cl}(\mathcal{U})\right) = 0. \quad \square\end{aligned}$$

Appendix B
Proof of Lemma 2

First, notice that, by construction, if $0 \notin \mathrm{conv}\,(\mathcal{V}(Q,\omega))$ for all $\omega \in \mathbb{R}$, then $0 \notin \mathcal{V}(Q,\omega)$. This implies that, if $\mathcal{O}(Q)$ is either TRUE or FALSE then the number of roots of $p(s,q)$ with nonnegative real part is invariant for all $q \in Q$. Hence, $p(s,q)$ is stable for all $q \in Q$ if $\mathcal{O}(Q) = \text{TRUE}$ and unstable for all $q \in Q$ if $\mathcal{O}(Q) = \text{FALSE}$. Therefore, the oracle above satisfies parts i) and ii) of Definition 2.

To show that the oracle satisfies part iii) of Definition 2 note the following. Define the set

$$\mathcal{U} \doteq \{q \in Q : p(s,q) \text{ has at least one root on the imaginary axis}\}.$$

Notice that, in this case, the set $\mathcal{U}$ in part iii) of Definition 2 is the union of the boundaries of the inertia invariant regions. Recall that, by inertia invariant regions we mean the sets $\mathcal{I} \subseteq Q$ for which the number of roots of $p(s,q)$ in the open right half plane is the same for all $q \in \mathcal{I}$. Given the assumptions made, the set $\mathcal{U}$ satisfies

$$\mathrm{vol}[\mathrm{cl}(\mathcal{U})] = 0.$$

Since the set Q is bounded and the polynomial is monic, there exists $\omega_{\max}$ satisfying

$$0 \in \mathrm{conv}(\mathcal{V}(Q,\omega)) \Rightarrow -\omega_{\max} \le \omega \le \omega_{\max}.$$

Now, given a rectangle Q, let $q_c(Q)$ be its center. Moreover, given ε, let $\delta(\varepsilon)$ be the minimum δ that satisfies the following condition: Given any hyperrectangle $Q \subseteq \mathcal{Q}$ such that $\mathrm{size}(Q) < \varepsilon$, then

$$\mathrm{conv}\,(\mathcal{V}(Q,\omega)) \subseteq B_\delta\,(p(j\omega, q_c(Q)))\; \forall \omega \in [-\omega_{\max}, \omega_{\max}]$$

where $B_\delta(z_0) = \{z : |z - z_0| \le \delta\}$. Given the fact that $p(j\omega, q)$ is a continuous function of ω and q, and that we are only considering bounded values of ω and q, we have that

$$\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0.$$

Now take q^* such that $p(s,q^*)$ does not have any zero on the imaginary axis and let

$$v^* = \min_{\omega \in [-\omega_{\max}, \omega_{\max}]} |p(j\omega, q^*)| > 0.$$

Then, there exists ε^* such that for all $\varepsilon \le \varepsilon^*$, $\delta(\varepsilon) < v^*/2$. This being the case and given the definitions above we have the following: Take any hyperectangle $Q \subseteq \mathcal{Q}$ such that $\mathrm{size}(Q) < \varepsilon^*$ and that

$$\mathcal{O}(Q) = \text{UNDECIDED} \Leftrightarrow 0 \in \mathrm{conv}\,(\mathcal{V}(Q,\omega^*))$$

for some frequency $\omega^* \in [-\omega_{\max}, \omega_{\max}]$. Now, since $0 \in \mathrm{conv}\,(\mathcal{V}(Q,\omega^*))$, we have

$$|p(j\omega^*, q)| \le 2\delta(\varepsilon) < v^*$$

for all $q \in Q$ which implies that $p(j\omega^*, q^*) \notin \mathcal{V}(Q,\omega^*)$ and, hence, $q^* \notin Q$. We conclude that part iii) of Definition 2 is satisfied. $\square$

References

[1] E.-W. Bai, R. Tempo, and M. Fu, "Worst-case properties of the uniform distribution and randomized algorithms for robustness analysis," *Math. Control, Signals, Syst.*, vol. 11, pp. 183–196, 1998.

[2] B. R. Barmish, *New Tools for Robustness of Linear Systems*. New York: MacMillan, 1994.

[3] B. R. Barmish and C. M. Lagoa, "The uniform distribution: A rigorous justification for its use in robustness analysis," *Math. Control, Signals, Syst.*, vol. 10, pp. 203–222, 1997.

[4] V. D. Blondel and J. N. Tsitsiklis, "A survey of computational complexity results in systems and control," *Automatica*, vol. 36, pp. 1249–1274, 2000.

[5] N. K. Bose and Y. Shi, "A simple general proof of Kharitonov's generalized stability criterion," *IEEE Trans. Circuits Syst.*, vol. 34, no. 10, pp. 1245–1247, 1997.

[6] R. K. Brayton, A. J. Hoffman, and T. R. Scott, "A thorem on inverses of convex sets of real matrices with application to the worst-case DC problem," *IEEE Trans. Circuits Syst.*, vol. 24, no. 8, pp. 409–415, 1977.

[7] *Probabilistic and Randomized Methods for Design Under Uncertainty*, G. Calafiore and F. Dabbene, Eds. London, U.K.: Springer-Verlag, 2006.

[8] S. W. Director and G. D. Hachtel, "The simplicial approximation approach to design centering," *IEEE Trans. Circuits Syst.*, vol. 24, no. 7, pp. 363–372, 1977.

[9] S. W. Director and L. M. Vidigal, "Statistical circuit design: A somewhat biased survey," in *Proc. Eur. Conf. Circuit Theory Design*, 1981, pp. 15–24.

[10] T. E. Djaferis and C. V. Hollot, "The stability of a family of polynomials can be deduced from a finite number $0(k^3)$ of frequency checks," *IEEE Trans. Autom. Control*, vol. 34, no. 9, pp. 1982–1986, 1989.

[11] S. G. Duvall, "A practical methodology for the statistical design of complex logic products for performance," *IEEE Trans. Very Large Integr. (VLSI) Syst.*, vol. 3, no. 1, pp. 12–123, 1995.

[12] M. E. Dyer and A. M. Frieze, "The complexity of computing the volume of a polyhedron," *SIAM J. Comput.*, vol. 17, pp. 967–874, 1988.

[13] A. Fettweis and S. Basu, "New results on stable multidimensional polynomials-Part I: Continuous case," *IEEE Trans. Circuits Syst.*, vol. 34, no. 10, pp. 1221–1232, 1987.

[14] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*. New York: Wiley, 1984.

[15] R. De Gaston and M. G. Safonov, "Exact calculation of the multiloop stability margin," *IEEE Trans. Autom. Control*, vol. 33, no. 2, pp. 156–171, 1988.

[16] W. Hoeffding, "Probability inequalities for sums of bounded random variables," *J. Amer. Statist. Assoc.*, vol. 58, pp. 13–30, 1963.

[17] H. Kettani and B. R. Barmish, "A new Monte Carlo circuit simulation paradigm with specific results for resistive networks," *IEEE Trans. Circuits Syst. I*, vol. 52, no. 6, pp. 1289–1299, 2006.

[18] V. L. Kharitonov, "Asymptotic stability of an equilibrium position of a family of systems of linear differential equations," (in Russian) *Differentsial'nye Uravneniya*, vol. 14, pp. 2086–2088, 1978.

[19] F. O'Doherty and J. P. Gleeson, "Phase diffusion coefficient for oscillators perturbed by colored noise," *IEEE Trans. Circuits Syst. II*, vol. 54, no. 5, pp. 435–439, 2007.

[20] H. Tang, "Symbolic statistical analysis of SNR variation for Delta-Sigma modulators," *IEEE Trans. Circuits Syst. II*, vol. 54, no. 8, pp. 720–724, 2007.

[21] R. Tempo, G. Calafiore, and F. Dabbene, *Randomized Algorithms for Analysis and Control of Uncertain Systems*. London, U.K.: Springer-Verlag, 2005, Commun. Control Eng. Series.

[22] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River, NJ: Prentice-Hall, 1996.

[23] X. Zhu, Y. Huang, and J. C. Doyle, "Soft vs. hard bounds in probabilistic robustness analysis," in *Proc. IEEE Conf. Decision Control*, 1996.

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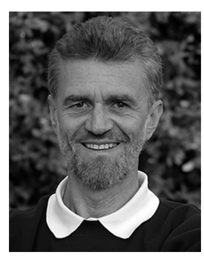

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After a period spent with the Dipartimento di Automatica e Informatica, Politecnico di Torino, he joined the IEIIT, National Research Council of Italy (CNR), Torino, where he is a Director of Research of Systems and Computer Engineering since 1991. He has held visiting and research positions with Kyoto University, the University of Illinois at Urbana-Champaign, the German Aerospace Research Organization in Oberpfaffenhofen, and Columbia University, New York. His research activities are mainly focused on complex systems with uncertainty, and related applications. He is the author or coauthor of more than 150 research papers published in international journals, books, and conferences. He is also a coauthor of the book *Randomized Algorithms for Analysis and Control of Uncertain Systems* (New York: Springer-Verlag, 2005).

Dr. Tempo has given plenary and semiplenary lectures at various conferences and workshops, including the European Control Conference, Kos, Greece, 2007, and the Robust Control Workshop, Delft, The Netherlands, 2005. He received the Outstanding Paper Prize Award from the International Federation of Automatic Control (IFAC) for a paper published in *Automatica*, and the Distinguished Member Award from the IEEE Control Systems Society. He is currently an Editor and Deputy Editor-in-Chief of Automatica. He is also Editor for Technical Notes and Correspondence of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL. He has served as a member of the Program Committee of several IEEE, IEE, IFAC, and European Union Control Association (EUCA) conferences, and as Program Chair of the first joint IEEE Conference on Decision and Control and European Control Conference, Seville, Spain, 2005. He has been Vice President for Conference Activities of the IEEE Control Systems Society during 2002–2003 and a member of the EUCA Council in 1998–2003. He is a Fellow of the IFAC.